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presented:

Edge-coloring of 3-uniform hypergraphs

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http://dx.doi.org/10.1016/j.dam.2016.06.009

Definitions

- Let $H = (V, E)$ be a *hypergraph*, $V(H)$ is a set of vertices, $E(H)$ is a multiset of non-empty subsets of $V(H)$ called *hyperedges(edges)*.
- Edge e and vertex v are *incident* if $v \in e$. Two edges e, e' are *adjacent* if they share a common vertex.
- $\Psi(e) = |e|$ denotes the edge cardinality. $\Psi(H) = \max_{e \in E(H)} \Psi(e)$ denotes the maximum cardinality of an edge in H.
- For a vertex $v \in V$, *degree* $deg(v)$ is a number of edges to which v is incident. $\Delta(H) = \max_{v \in V(H)} deg(v)$ is a *degree of* H.
- Hypergraph H is $d-$*uniform* if $\forall e \in E(H), \Psi(e) = d$.
- A *proper edge-coloring* of a hypergraph H with k colors is a function $c : E(H) \to \{1, \ldots, k\}$ such that no two adjacent edges are assigned the same color.
- The *chromatic index* $\chi'(H)$ of H is a number of colors in an optimal (minimal) edge-coloring of H.
- A *line graph* $L(H)$ of hypergraph H is a simple graph where vertices represent hyperedges of H and two vertices in $L(H)$ are adjacent if and only if their corresponding hyperedges are adjacent.
- A graph G is an *underlying (host) graph* of hypergraph H if $V(G) = V(E)$ and each edge $e \in E(H)$ induces a connected subgraph in G.
- A hypergraph H is called a *hypertree/hypercycle/hypercactus* if there exists a *tree/cycle/cactus* which is an underlying graph for H.

Theorems

Fact 1 *For any 3-uniform hypergraph H the following holds: $\Delta(H) \le \chi'(H) \le 3\Delta(H) - 2$.*

The edge-coloring of hypergraph H is equivalent to vertex-coloring of $L(H)$, thus the above fact can be generalized to $\chi'(H) \le \Delta(L(H)) + 1$, or due to Brooks' theorem to $\chi'(H) \le \Delta(L(H))$ unless $L(H)$ is a complete graph or an odd cycle.

Fact 2 *Let H be hypertree. Then it can be edge-colored in polynomial time.*

It can be done using a modified version of BFS (breadth-first search).

Lemma 3 *Let H be hypercycle with m edges and $\Psi(H) = 3$. Then it can be edge-colored in $O(m^{3/2})$.*

It is done by transforming the graph into a proper circular arc graph which can be colored using Teng and Tucker's approach in $O(n^{3/2})$ [1]. Teng and Tucker's work is a refinement of an original $O(n^2)$ algorithm by Orlin et al. [2].

Theorem 4 *Edge-coloring of a 3-uniform hypercactus with m edges can be done in time $O(m^{3/2})$.*

Using results for hypertrees and hypercycles, we devise a recursive procedure to color the whole hypercactus.

Theorem 5 *It is NP-complete to decide whether a 3-partite hypergraph of degree 3 is 3-edge-colorable.*

The reduction is done from the problem of edge precoloring extension to proper 3-edge-coloring for bipartite graphs of degree 3 precolored with at most 3 colors, which has been proven to be NP-complete by Fiala [3], who in turn used a nice reduction from Not All Equal 3-SAT (NAE-3-SAT).

References

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