Spectral bounds for 1D discrete Schrödinger and Dirac operators with complex potentials

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Contents

1 The discrete Schrödinger operator

2 The discrete Dirac operator
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Definitions

- Let \( \{e_n\}_{n \in \mathbb{Z}} \) stands for the standard basis of \( \ell^2(\mathbb{Z}) \).
- The discrete Laplacian:

\[
H_0 e_n = e_{n-1} + e_{n+1}, \quad \forall n \in \mathbb{Z}.
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  \[
  V e_n := \nu_n e_n, \quad \forall n \in \mathbb{Z}.
  \]
- The discrete Schrödinger operator: \( H_V = H_0 + V \),
  \[
  H_V = \begin{pmatrix}
  \ddots & \ddots & \ddots & & \\
  & 1 & \nu_{-1} & 1 & \\
  & 1 & \nu_0 & 1 & \\
  & 1 & \nu_1 & 1 & \\
  & & & & \ddots & \ddots & \ddots
  \end{pmatrix}.
  \]
Basic facts

- One has

\[ \sigma(H_0) = \sigma_{\text{ess}}(H_0) = [-2, 2]. \]
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- If \( v_n \to 0 \), as \( n \to \pm\infty \), then \( V \) is compact and \( \sigma_{ess}(H_V) = [-2, 2] \).
- The resolvent of \( H_0 \):
  \[ (H_0 - \lambda)^{-1}_{m,n} = \frac{k^{m-n}}{k - k^{-1}}, \quad \forall m, n \in \mathbb{Z}, \]
  where \( \lambda = k^{-1} + k \).
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  where \( \lambda = k^{-1} + k \).

- The Joukowski map:
  \[ \lambda(k) = k^{-1} + k \]
  is 1–1 mapping of the punctured unit disk \( 0 < |k| < 1 \) onto \( \mathbb{C} \setminus [-2, 2] \).
\(\ell^1\)-potentials

**Theorem (\(\ell^1\)-potential)**

Let \(\nu \in \ell^1(\mathbb{Z})\). Then

\[
\sigma_p(H_\nu) \subset \left\{ \lambda \in \mathbb{C} \setminus (-2, 2) \mid |\lambda^2 - 4| \leq \|\nu\|_{\ell^1(\mathbb{Z})}^2 \right\}.
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In addition, the estimate is **optimal** in the following sense:
The discrete Schrödinger operator

\( \ell^1 \)-potentials

Theorem (\( \ell^1 \)-potential)

Let \( \nu \in \ell^1(\mathbb{Z}) \). Then

\[
\sigma_p(H_V) \subset \left\{ \lambda \in \mathbb{C} \setminus (-2, 2) \mid |\lambda^2 - 4| \leq \| \nu \|_{\ell^1(\mathbb{Z})}^2 \right\}.
\]

In addition, the estimate is optimal in the following sense:

To any boundary point of the spectral enclosure which does not belong to \((-2, 2)\), there exists an \( \ell^1 \)-potential \( V \) so that this boundary point is an eigenvalue of the corresponding discrete Schrödinger operator \( H_V \).
The discrete Schrödinger operator

Geometry of the boundary curve

The boundary curve for $Q := \|u\|_{\ell^1(Z)}$:

$$|\lambda^2 - 4| = Q^2.$$
Proof

The goal is to prove:

\[ \sigma_p(H_V) \subset \left\{ \lambda \in \mathbb{C} \setminus (-2, 2) \mid |\lambda^2 - 4| \leq \|v\|_{\ell^1(\mathbb{Z})}^2 \right\}. \]
The discrete Schrödinger operator

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One has $v \in \ell^1(\mathbb{Z}) \implies (-2, 2) \cap \sigma_p(H_V) = \emptyset$ (Jost solution).
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The points \( \pm 2 \) are always included in the spectral enclosure.
The discrete Schrödinger operator

Proof

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- The points \( \pm 2 \) are always included in the spectral enclosure.
- For \( \lambda \notin [-2, 2] \equiv \sigma(H_0) \), the proof relies on the Birman–Schwinger principle (one implication):
  \[ \lambda \in \sigma_p(H_V) \implies -1 \in \sigma_p(K(\lambda)), \]
  for
  \[ K(\lambda) := |V|^{1/2} (H_0 - \lambda)^{-1} V_{1/2}, \]
  and
  \[ |V|^{1/2} e_n = \sqrt{|v_n|} e_n \quad \text{and} \quad V_{1/2} e_n = \text{sgn}(v_n) \sqrt{|v_n|} e_n \]
  with the complex signum function \( \text{sgn} z = z/|z| \), if \( z \neq 0 \), and \( \text{sgn} 0 = 0 \).
Proof

- The goal is to prove:

$$\sigma_p(H_V) \subset \left\{ \lambda \in \mathbb{C} \setminus (-2, 2) \mid |\lambda^2 - 4| \leq \|\nu\|_{\ell^1(\mathbb{Z})}^2 \right\}.$$

- One has $\nu \in \ell^1(\mathbb{Z}) \implies (-2, 2) \cap \sigma_p(H_V) = \emptyset$ (Jost solution).
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- For $\lambda \notin [-2, 2] \equiv \sigma(H_0)$, the proof relies on the Birman–Schwinger principle (one implication):

$$\lambda \in \sigma_p(H_V) \implies -1 \in \sigma_p(K(\lambda)),$$

for

$$K(\lambda) := |V|^{1/2} (H_0 - \lambda)^{-1} V_{1/2},$$

and

$$|V|^{1/2} e_n = \sqrt{|\nu_n|} e_n \quad \text{and} \quad V_{1/2} e_n = \text{sgn}(\nu_n) \sqrt{|\nu_n|} e_n$$

with the complex signum function $\text{sgn} \, z = z/|z|$, if $z \neq 0$, and $\text{sgn} \, 0 = 0$.
- In particular,

$$\lambda \in \sigma_p(H_V) \implies \|K(\lambda)\| \geq 1.$$
Proof - the part based on the Birman–Schwinger principle

Let $\lambda \notin [-2, 2] \equiv \sigma(H_0)$. 
Proof - the part based on the Birman–Schwinger principle

- Let \( \lambda \notin [-2, 2] \equiv \sigma(H_0) \).
- Then \( \lambda = k^{-1} + k \) with \( |k| < 1 \) and one has
  \[
  \left| (H_0 - \lambda)^{-1} \right|_{m,n} = \frac{|k|^{m-n}}{|k - k^{-1}|} \leq \frac{1}{|k - k^{-1}|} = \frac{1}{\sqrt{|\lambda^2 - 4|}}, \quad \forall m, n \in \mathbb{Z}.
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- For any $\psi \in \ell^2(\mathbb{Z})$, we estimate
  
  $$\|K(\lambda)\psi\|_{\ell^2(\mathbb{Z})}^2 \leq \sum_{m \in \mathbb{Z}} \left( \sum_{n \in \mathbb{Z}} \sqrt{|u_m|} \left|(H_0 - \lambda)_{m,n}^{-1}\right| \sqrt{|v_n|} |\psi_n| \right)^2$$
  
  $$\leq \frac{\|v\|_{\ell^1(\mathbb{Z})}^2}{|\lambda^2 - 4|} \left( \sum_{m \in \mathbb{Z}} \sqrt{|u_m|} |\psi_n| \right)^2 \leq \frac{\|v\|_{\ell^1(\mathbb{Z})}^2}{|\lambda^2 - 4|} \|\psi\|_{\ell^2(\mathbb{Z})}^2.$$
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For any \( \psi \in \ell^2(\mathbb{Z}) \), we estimate

\[
\| K(\lambda) \psi \|_{\ell^2(\mathbb{Z})}^2 \leq \sum_{m \in \mathbb{Z}} \left( \sum_{n \in \mathbb{Z}} \sqrt{|v_m|} \left| (H_0 - \lambda)^{-1}_{m,n} \right| \sqrt{|v_n|} |\psi_n| \right)^2 \leq \frac{\| v \|_{\ell^1(\mathbb{Z})}^2}{|\lambda^2 - 4|} \| \psi \|_{\ell^2(\mathbb{Z})}^2.
\]

In other words,

\[
\| K(\lambda) \|^2 \leq \frac{\| v \|_{\ell^1(\mathbb{Z})}^2}{|\lambda^2 - 4|}
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\leq \frac{\|v\|_{\ell^1(\mathbb{Z})}}{|\lambda^2 - 4|} \left( \sum_{m \in \mathbb{Z}} \sqrt{|v_n|} |\psi_n| \right)^2 \leq \frac{\|v\|_{\ell^1(\mathbb{Z})}^2}{|\lambda^2 - 4|} \|\psi\|_{\ell^2(\mathbb{Z})}^2.
\]

In other words,

\[
\|K(\lambda)\|^2 \leq \frac{\|v\|_{\ell^1(\mathbb{Z})}^2}{|\lambda^2 - 4|}
\]

Thus, if \( \lambda \in \sigma_p(H_\nu) \), then

\[
|\lambda^2 - 4| \leq \|v\|_{\ell^1(\mathbb{Z})}^2.
\]
Many works make use of the Birman–Swinger principle...

Many various spectral bounds (mainly) for differential operators such as Schrödinger and Dirac operators were obtained by applying the Birman–Schwinger principle.
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An incomplete list of authors:

Abramov, Aslanyan, Behrndt, Cuenin, Davies, Enblom, Frank, Fanelli, Ibrogimov, Krejčířík, Langer, Laptev, Lee, Lieb, Lotoreichik, Rohleder, Safronov, Seiringer, Seo, Tretter, Vega,...
The optimality

- Delta potential:

\[ v_n := \omega \delta_{n,0}, \quad \forall n \in \mathbb{Z}, \]

where \( \omega \in \mathbb{C} \) is a coupling constant.
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- For \( \omega \in \mathbb{C} \setminus [-2i, 2i] \),
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  \[ |\lambda_\omega^2 - 4| = |\omega|^2 \equiv \|v\|^2_{l^1(\mathbb{Z})}. \]
The discrete Schrödinger operator

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- The eigenvalue \( \lambda_\omega \) lies on the boundary curve of the spectral enclosure because
  \[ |\lambda_\omega^2 - 4| = |\omega|^2 \equiv \|v\|_{\ell^1(\mathbb{Z})}^2. \]

- Moreover, for any \( Q > 0 \), one has
  \[ \{ \lambda_\omega \mid \omega = Q e^{i\theta}, -\pi < \theta \leq \pi \} = \{ \lambda \in \mathbb{C} \mid |\lambda^2 - 4| = Q^2 \}. \]
Numerical illustration: the delta potential demonstrates optimality
The discrete Schrödinger operator

$\ell^p$-potentials, $p > 1$

**Theorem ($\ell^p$-potential)**

Let $1 < p \leq \infty$ and $v \in \ell^p(\mathbb{Z})$. Denote by $q \in [1, \infty)$ the corresponding Hölder exponent, i.e.,

$$\frac{1}{p} + \frac{1}{q} = 1.$$ 

Then

$$\sigma(H_v) \subset \left\{ k + \frac{1}{k} \mid k \in \mathbb{C} \setminus \{0\}, \ |k| \leq 1 \text{ and } \left| k - \frac{1}{k} \right| \left( \frac{1 - |k|^q}{1 + |k|^q} \right)^{1/q} \leq \|v\|_{\ell^p(\mathbb{Z})} \right\}.$$ 

**Remarks:**

The proof is based again on Birman–Schwinger principle and uses either the Schur test or discrete Young’s inequality. No optimality result. The interval $[-2, 2]$ always involved in the spectral enclosure $\Rightarrow$ no consequences for embedded eigenvalues.
The discrete Schrödinger operator

\( \ell^p \)-potentials, \( p > 1 \)

**Theorem (\( \ell^p \)-potential)**

Let \( 1 < p \leq \infty \) and \( \nu \in \ell^p(\mathbb{Z}) \). Denote by \( q \in [1, \infty) \) the corresponding Hölder exponent, i.e.,

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\sigma(H_\nu) \subset \left\{ k + \frac{1}{k} \mid k \in \mathbb{C} \setminus \{0\}, |k| \leq 1 \text{ and } \left| k - \frac{1}{k} \right| \left( \frac{1 - |k|^q}{1 + |k|^q} \right)^{1/q} \leq \|\nu\|_{\ell^p(\mathbb{Z})} \right\}.
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\textbf{\(\ell^p\)-potentials, \(p > 1\)}

\section*{Theorem (\(\ell^p\)-potential)}

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\frac{1}{p} + \frac{1}{q} = 1.
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\sigma(H\nu) \subset \left\{ k + \frac{1}{k} \mid k \in \mathbb{C} \setminus \{0\}, \ |k| \leq 1 \ \text{and} \ \left| k - \frac{1}{k} \right| \left( \frac{1 - |k|^q}{1 + |k|^q} \right)^{1/q} \leq \|\nu\|_{\ell^p(\mathbb{Z})} \right\}.
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- The proof is based again on Birman–Schwinger principle and uses either the Schur test or discrete Young’s inequality.
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$\ell^p$-potentials: plots of the spectral enclosure for $p = 2$.
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- Let \( \{ e_n \}_{n \in \mathbb{Z}} \) stands for the standard basis of \( \ell^2(\mathbb{Z}) \).
- The operator \( d : \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z}) : \)
  \[
  de_n := e_n - e_{n+1}, \quad \forall n \in \mathbb{Z}.
  \]
The discrete Dirac operator

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d e_n := e_n - e_{n+1}, \quad \forall n \in \mathbb{Z}.
  \]
- Free discrete Dirac operator \( D_0 \):
  \[
  D_0 := \begin{pmatrix} m & d \\ d^* & -m \end{pmatrix}
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  acting on \( \ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z}) \), where \( m \geq 0 \) and \( d^* \) is the adjoint operator to \( d \).
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  acting on \( \ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z}) \), where \( m \geq 0 \) and \( d^* \) is the adjoint operator to \( d \).
- Considered potentials:
  \[
  V = \begin{pmatrix} V^{1,1} & V^{1,2} \\ V^{2,1} & V^{2,2} \end{pmatrix},
  \]
  where \( V^{i,j} \) act on \( \ell^2(\mathbb{Z}) \) as diagonal operators determined by doubly infinite complex sequences.
By using a suitable orthonormal basis of $\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$, $D_0$ can be represented by the $2 \times 2$-block tridiagonal Laurent matrix:

$$D_0 = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots \\ \vdots & a^T & b & a \\ a^T & b & a \\ \vdots & a^T & b & a \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix},$$

where

$$b := \begin{pmatrix} -m & 1 \\ 1 & m \end{pmatrix} \quad \text{and} \quad a := \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$
By using a suitable orthonormal basis of $\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$, $D_0$ can be represented by the $2 \times 2$-block tridiagonal Laurent matrix:

$$D_0 = \begin{pmatrix} \cdots & b & a & \cdots \\ a^T & b & a & a^T \\ \cdots & b & a & \cdots \\ a^T & b & a & \cdots \end{pmatrix},$$

where

$$b := \begin{pmatrix} -m & 1 \\ 1 & m \end{pmatrix} \quad \text{and} \quad a := \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

The considered perturbation of $D_0$:

$$V = \bigoplus_{n \in \mathbb{Z}} \nu_n, \quad \text{where} \quad \nu_n := \begin{pmatrix} \nu_{11} & \nu_{12} \\ \nu_{21} & \nu_{22} \end{pmatrix}.$$
Facts about $D_0$

- The spectrum:

$$\sigma(D_0) = \sigma_{\text{ess}}(D_0) = \left[ -\sqrt{m^2 + 4}, -m \right] \cup \left[ m, \sqrt{m^2 + 4} \right].$$
Facts about $D_0$

- The spectrum:
  \[ \sigma(D_0) = \sigma_{\text{ess}}(D_0) = [-\sqrt{m^2 + 4}, -m] \cup [m, \sqrt{m^2 + 4}] . \]

- An important correspondence: The equation
  \[ \lambda^2 = m^2 + 2 - k - k^{-1} \]
  determines a one-to-two mapping $\lambda = \lambda(k)$ from $0 < |k| < 1$ onto $\rho(D_0)$. 

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Facts about $D_0$

- The spectrum:
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- An important correspondence: The equation
  \[ \lambda^2 = m^2 + 2 - k - k^{-1} \]
determines a one-to-two mapping $\lambda = \lambda(k)$ from $0 < |k| < 1$ onto $\rho(D_0)$.

- The $2 \times 2$-block Laurent matrix representation of the resolvent:
  \[ (D_0 - \lambda)^{-1}_{m,n} = T_{n-m}(k), \]
  where
  \[ T_0(k) = \frac{1}{k^{-1} - k} \begin{pmatrix} \lambda - m & 1 - k \\ 1 - k & \lambda + m \end{pmatrix}, \]
  \[ T_j(k) = T_{-j}^T(k) = \frac{k^j}{k^{-1} - k} \begin{pmatrix} \lambda - m & 1 - k \\ 1 - k^{-1} & \lambda + m \end{pmatrix}, \quad j \geq 1. \]
\( \ell^1 \)-potentials

**Theorem (\( \ell^1 \)-potential)**

Let \( V \in \ell^1(\mathbb{Z}, \mathbb{C}^{2 \times 2}) \). Then

\[
\sigma_p(D_V) \subset \left\{ \lambda \in \mathbb{C} \mid |\lambda^2 - m^2||\lambda^2 - m^2 - 4| \leq (|\lambda + m| + |\lambda - m|)^2 \| V \|_1^2 \right\}.
\]

Remark: The Banach space \( \ell^p(\mathbb{Z}, \mathbb{C}^{2 \times 2}) \) is equipped with the norm \( \| V \|_p = \left( \sum_{n \in \mathbb{Z}} |\nu_n|^p \right)^{1/p} \), if \( 1 \leq p < \infty \), \( \| V \|_\infty = \sup_{n \in \mathbb{Z}} |\nu_n| \), where \( |\nu_n| \) denotes the operator norm of the matrix \( \nu_n \in \mathbb{C}^{2 \times 2} \).
\( \ell^1 \)-potentials

**Theorem (\( \ell^1 \)-potential)**

Let \( V \in \ell^1(\mathbb{Z}, \mathbb{C}^{2\times 2}) \). Then

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\]

**Remark:** The Banach space \( \ell^p(\mathbb{Z}, \mathbb{C}^{2\times 2}) \) is equipped with the norm

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\|V\|_p = \left( \sum_{n \in \mathbb{Z}} |v_n|^p \right)^{1/p}, \quad \text{if } 1 \leq p < \infty, \quad \|V\|_\infty = \sup_{n \in \mathbb{Z}} |v_n|,
\]

where \( |v_n| \) denotes the operator norm of the matrix \( v_n \in \mathbb{C}^{2\times 2} \).
Geometry of the boundary curve for $m = 1$
Embedded eigenvalues

**Corollary:**

Let $V \in \ell^1(\mathbb{Z}, \mathbb{C}^{2 \times 2})$. If $2 \| V \|_1^2 < (m^2 + 2 - m\sqrt{m^2 + 4})$ then the union of two intervals

$$(-\tau_+, -\tau_-) \cup (\tau_-, \tau_+),$$

where

$$\tau_{\pm} = \sqrt{2 + m^2 - 2 \| V \|_1^2 \pm 2 \sqrt{1 - (m^2 + 2) \| V \|_1^2 + \| V \|_4^4}},$$

is free of embedded eigenvalues of $H_V$. 
Optimality

- The presented bound for $\ell^1$-potentials is not optimal.
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- The presented bound for $\ell^1$–potentials is not optimal.
- A tighter bound exists:

$$\sigma_p(D_V) \setminus \sigma(D_0) \subset \{ \lambda \in \mathbb{C} \setminus \sigma(D_0) \mid \max \{ |T_0(k)|, |T_1(k)| \|V\|_1 \geq 1 \},$$

where $k$ is the unique point in $\{ k \in \mathbb{C} | 0 < |k| < 1 \}$ such that $\lambda^2 = m^2 + 2 - k - k^{-1}$. The $2 \times 2$ complex matrices $T_0(k)$ and $T_1(k)$ appear in the formula for the resolvent $(D_0 - \lambda)^{-1}$. Their spectral norms can be expressed explicitly but lead to complicated expressions:

$$|T_1(k)|^2 = |k|^2 |\lambda + m|^2 + |\lambda - m|^2 + (|k| + |k|^{-1}) |\lambda^2 - m^2| \|\lambda^2 - m^2\|.$$
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- The presented bound for $\ell^1$–potentials is not optimal.
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**Theorem (improved spectral enclosure for $\ell^1$-potential)**

Let $V \in \ell^1(\mathbb{Z}, \mathbb{C}^{2 \times 2})$. Then

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$$|T_0(k)|^2 = \ldots \text{even more complicated :(.}$$
Optimality of the improved spectral enclosure for $\ell^1$-potential

We were able to prove only a “partial” optimality, i.e., only a part of the boundary of the spectral enclosure can be approached by an eigenvalue of $D_V$ for some $V \in \ell^1(\mathbb{Z}, \mathbb{C}^{2 \times 2})$. 
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We were able to prove only a “partial” optimality, i.e., only a part of the boundary of the spectral enclosure can be approached by an eigenvalue of $D_V$ for some $V \in \ell^1(\mathbb{Z}, \mathbb{C}^{2\times 2})$. 
Spectral enclosures for $\ell^p$–potentials, $p > 1$

**Theorem (spectral enclosures for $\ell^p$–potentials)**

Let $1 < p \leq \infty$, $q$ the Hölder dual index to $p$, and assume $V \in \ell^p(\mathbb{Z}, \mathbb{C}^{2\times2})$. Then $\sigma(D_V) \setminus \sigma(D_0)$ is a subset of:

1. A simpler bound:
   
   $$\{ \lambda \in \mathbb{C} \setminus \sigma(D_0) \mid \left| \lambda - m \right| + \left| \lambda + m \right| \leq \left( \frac{1}{k^1} + 2^{1/2} \left| k \right|^{1/q} + 2^{1/2} \left| k \right|^{1/q} \right)^{1/q} \left\| V \right\|_p \geq 1 \}.$$

2. A tighter bound:
   
   $$\{ \lambda \in \mathbb{C} \setminus \sigma(D_0) \mid \left( \left| T_0(k) \right|^{q} + 2 \right)^{1/q} \left( \left| T_1(k) \right|^{q} + 2 \right)^{1/q} \left\| V \right\|_p \geq 1 \}.$$
The discrete Dirac operator

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$$\left\{ \lambda \in \mathbb{C} \setminus \sigma(D_0) \left| \frac{|\lambda - m| + |\lambda + m|}{|k^{-1} - k|} \left(1 + \frac{2\sqrt{|k|^q}}{1 - |k|^q}\right)^{1/q} \|V\|_p \geq 1 \right. \right\}.$$
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Thank you!