The Moment Problem

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Outline

1. Motivation
2. What the moment problem is?
3. Existence and uniqueness of the solution - operator approach
4. Jacobi matrix and Orthogonal Polynomials
5. Sufficient conditions for determinacy
6. The set of solutions of indeterminate moment problem
Chebychev’s question: *If for some positive function* $f$,

$$\int_{\mathbb{R}} x^n f(x) dx = \int_{\mathbb{R}} x^n e^{-x^2} dx, \quad n = 0, 1, \ldots$$

*can we then conclude that* $f(x) = e^{-x^2}$?

*That is:* Is the normal density uniquely determined by its moment sequence?

*Answer:* yes in the sense that $f(x) = e^{-x^2}$ a.e. wrt Lebesgue measure on $\mathbb{R}$.

What happens if one replaces the normal density by something else? The general answer to the Chebychev’s question is no. Suppose, e.g., $X \sim N(0, \sigma^2)$ and consider densities of $\exp(X)$ (lognormal distribution) or $\sinh(X)$ then we lose the uniqueness.

A tough problem: What can be said when there is no longer uniqueness?
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What is moment problem

Let $I \subset \mathbb{R}$ be an open interval. For a positive measure $\mu$ on $I$ the $n$th moment is defined as

$$\int_I x^n d\mu(x), \quad (\text{provided the integral exists}).$$

Suppose a real sequence $\{s_n\}_{n \geq 0}$ is given. The moment problem on $I$ consists of solving the following three problems:

1. Does there exist a positive measure on $I$ with moments $\{s_n\}_{n \geq 0}$?
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2. is this positive measure uniquely determined by moments \( \{s_n\}_{n \geq 0} \)? (determinate case)
   If this is not the case,
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One can restrict oneself to cases:

- $I = \mathbb{R}$ - Hamburger moment problem ($\mathcal{M}_H = \text{set of solutions}$)
- $I = [0, +\infty)$ - Stieltjes moment problem ($\mathcal{M}_S = \text{set of solutions}$)
- $I = [0, 1]$ - Hausdorff moment problem
The moment problem has a solution on $[0, 1]$ iff sequence $\{s_n\}_{n \geq 0}$ is completely monotonic, i.e.,

$$(-1)^k (\Delta^k s)_n \geq 0$$

for all $k, n \in \mathbb{Z}_+$, where $(\Delta s)_n = s_{n+1} - s_n$. 

The Hausdorff moment problem is not interesting from the uniqueness of the solution point of view, since The Hausdorff moment problem is always determinate! 

Steps of the proof:
- measure with finite support is uniquely determined by its moments (Vandermonde matrix),
- approximation theorem of Weierstrass,
- Riesz representation theorem.

Consequently, we will further discuss the Stieltjes and Hamburger moment problem only.
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Consequently, we will further discuss the Stieltjes and Hamburger moment problem only.
Existence of the solution

For \( \{s_n\}_{n \geq 0} \), we denote \( H_N(s) \) the \( N \times N \) Hankel matrix with entries \((H_N(s))_{ij} := s_{i+j}, i, j \in \{0, 1, \ldots, N-1\}\).
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Define two sesquilinear forms \( H_N \) and \( S_N \) on \( \mathbb{C}^N \) by

\[
H_N(x, y) := \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \bar{x}_i y_j s_{i+j} \quad \text{and} \quad S_N(x, y) := \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \bar{x}_i y_j s_{i+j+1}.
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Hence \(H_N(x, y) = (x, H_N(s)y)\) and \(S_N(x, y) = (x, H_N(Ts)y)\) ((., .) Euclidean inner product).
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Let \( \mu \in \mathcal{M}_H \) or \( \mu \in \mathcal{M}_S \) with infinite support. By observing that

\[
H_N(y, y) = \int \left| \sum_{i=0}^{N-1} y_i x^i \right|^2 d\mu(x) \quad \text{and} \quad S_N(y, y) = \int x \left| \sum_{i=0}^{N-1} y_i x^i \right|^2 d\mu(x),
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one immediately gets the following.
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### Necessary condition for the existence

A necessary condition for the Hamburger moment problem to have a solution (with infinite support) is the sesquilinear form \( H_N \) is PD for all \( N \in \mathbb{Z}_+ \). A necessary condition for the Stieltjes moment problem to have a solution (with infinite support) is both sesquilinear forms \( H_N \) and \( S_N \) are PD for all \( N \in \mathbb{Z}_+ \).
Existence of the solution

Let $H_N$ be PD for all $N \in \mathbb{N}$. 

Let $C[x]$ be the ring of complex polynomials. 

For $P, Q \in C[x]$, define positive definite inner product 

$$\langle P, Q \rangle := H_N(a, b).$$ 

By using standard procedure, we can complete $C[x]$ to a Hilbert space $H(s)$. 

Define densely defined operator $A$ on $H(s)$ with $\text{Dom}(A) = C[x]$ by 

$$A[P(x)] = xP(x).$$ 

Since $\langle P, A[Q] \rangle = S_N(a, b) = \langle A[P], Q \rangle$, $A$ is a symmetric operator. 

Especially, $\langle 1, A^n 1 \rangle = s_n$, $n \in \mathbb{N}$. 

Existence of the solution

- Let $H_N$ be PD for all $N \in \mathbb{N}$.
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Existence of the solution

- Let $H_N$ be PD for all $N \in \mathbb{N}$.
- Let $\mathbb{C}[x]$ be the ring of complex polynomials.
- For $P, Q \in \mathbb{C}[x]$,

$$P(x) = \sum_{k=0}^{N-1} a_k x^k, \quad \text{and} \quad Q(x) = \sum_{k=0}^{N-1} b_k x^k,$$

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- By using standard procedure, we can complete $\mathbb{C}[x]$ to a Hilbert space $\mathcal{H}^{(s)}$. 

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- By using standard procedure, we can complete $\mathbb{C}[x]$ to a Hilbert space $\mathcal{H}^{(s)}$.

Define densely defined operator $A$ on $\mathcal{H}^{(s)}$ with $\text{Dom}(A) = \mathbb{C}[x]$ by
\[ A[P(x)] = xP(x). \]
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  $A$ is a symmetric operator.
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- Let $\mathbb{C}[x]$ be the ring of complex polynomials.
- For $P, Q \in \mathbb{C}[x],
  \begin{align*}
P(x) &= \sum_{k=0}^{N-1} a_k x^k, \quad \text{and} \quad Q(x) = \sum_{k=0}^{N-1} b_k x^k,
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- Since
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  $A$ is a symmetric operator.
- Especially,
  \[ \langle 1, A^n 1 \rangle = s_n, \quad n \in \mathbb{N}. \]
A has a self-adjoint extension since it commutes with a complex conjugation operator $C$ on $\mathbb{C}[x]$ (von Neumann).
Existence of the solution

- $A$ has a self-adjoint extension since it commutes with a complex conjugation operator $C$ on $\mathbb{C}[x]$ (von Neumann).
- If each $S_N$ is PD, then
  \[
  \langle P, A[P] \rangle = S_N(a, a) \geq 0, \quad \text{for all } P \in \mathbb{C}[x],
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  and it follows $A$ has a non-negative self-adjoint extension $A_F$, the Friedrichs extension.
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- Let \( A' \) be a self-adjoint extension of \( A \). By the spectral theorem there is a projection valued spectral measure \( E_{A'} \) and positive measure

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$$\langle 1, f(A')1 \rangle = \int_{\mathbb{R}} f(x)d\mu(x).$$
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Hence, for a suitable function $f$, it holds

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Especially, for $f(x) = x^n$, one finds

$$s_n = \langle 1, A^n1 \rangle = \langle 1, (A')^n1 \rangle = \int_{\mathbb{R}} x^n d\mu(x),$$

since $\text{Dom}(A^n) \subset \text{Dom}((A')^n)$. 
Existence of the solution

We see a self-adjoint extension of $A$ yields a solution of the Hamburger moment problem.

Hence we arrive at the theorem on the existence of the solution.

Theorem (Existence)

i) A necessary and sufficient condition for $M_H \neq \emptyset$ (with infinite support) is 
$$\det H_N(s) > 0$$
for all $N \in \mathbb{N}$.

ii) A necessary and sufficient condition for $M_S \neq \emptyset$ (with infinite support) is 
$$\det H_N(s) > 0 \land \det S_N(s) > 0$$
for all $N \in \mathbb{N}$.

Historically, this result has not been obtained by using the spectral theorem that was invented later.
Existence of the solution

- We see a self-adjoint extension of $A$ yields a solution of the Hamburger moment problem.
- Moreover, a non-negative self-adjoint extension has $\text{supp}(\mu) \subset [0, \infty)$ and so yields a solution of the Stieltjes moment problem.
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In view of the connection of the moment problem and self-adjoint extensions, the following result is reasonable.

Theorem (Uniqueness)

i) A necessary and sufficient condition for the Hamburger moment problem to be determinate is that the operator $\mathcal{A}$ is essentially self-adjoint (i.e., it has a unique self-adjoint extension).

ii) A necessary and sufficient condition for the Stieltjes moment problem to be determinate is that the operator $\mathcal{A}$ has a unique non-negative self-adjoint extension.

It is not easy to prove the theorem. In one direction, it is not clear that distinct self-adjoint extensions $\mathcal{A}_1'$ and $\mathcal{A}_2'$ give rise to distinct measures $\mu_1$ and $\mu_2$.

The other direction is even less clear. For not only is it not obvious, it is false that every solution of the moment problem arises from some measure given by spectral measure of some self-adjoint extension.

A solution of the moment problem which comes from a self-adjoint extension of $\mathcal{A}$ is called N-extremal solution (von Neumann [Simon], extremal [Shohat-Tamarkin]).
In view of the connection of the moment problem and self-adjoint extensions, the following result is reasonable.

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i) A necessary and sufficient condition for the Hamburger moment problem to be determinate is that the operator $A$ is essentially self-adjoint (i.e., it has a unique self-adjoint extension).

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A solution of the moment problem which comes from a self-adjoint extension of $A$ is called *N-extremal* solution (von Neumann [Simon], extremal [Shohat-Tamarkin]).
Consider set \( \{1, x, x^2, \ldots \} \subset \mathcal{H}^{(s)} \) which is linearly independent \((H_N \text{ PD})\) and span \( \mathcal{H}^{(s)} \).
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By construction, \( P_n \) is a polynomial of degree \( n \) with real coefficients and

\[
\langle P_m, P_n \rangle = \delta_{mn}
\]

for all \( m, n \in \mathbb{Z}_+ \). These are well-known Orthogonal Polynomials.
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\( \{P_n\}_{n=0}^{\infty} \) are determined by moment sequence \( \{s_n\}_{s=0}^{\infty} \),

\[
P_n(x) = \frac{1}{\sqrt{\det[H_{n+1}(s)H_n(s)]}} \begin{vmatrix} s_0 & s_1 & \ldots & s_n \\ s_1 & s_2 & \ldots & s_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n-1} & s_n & \ldots & s_{2n-1} \\ 1 & x & \ldots & x^n \end{vmatrix}
\]
Since \( \text{span}(1, x, \ldots, x^n) = \text{span}(P_0, P_1, \ldots, P_n) \), \( xP_n(x) \) has an expansion in \( P_0, P_1, \ldots, P_{n+1} \).
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Moreover, if \( 0 \leq j < n - 1 \), one has

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There are sequences \( \{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}, \text{ and } \{c_n\}_{n=0}^{\infty} \) such that

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xP_n(x) = c_n P_{n+1}(x) + b_n P_n(x) + a_{n-1} P_{n-1}(x), \quad (P_{-1}(x) := 0),
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Hence \( A \) has, in the basis \( \{P_n\}_{n=0}^{\infty} \), has tridiagonal matrix representation and \( \text{Dom}(A) \) is the set of sequences of finite support.
The realization of elements of $\mathcal{H}^{(s)}$ as $\sum_{n=0}^{\infty} \lambda_n P_n$, with $\sum_{n=0}^{\infty} |\lambda_n|^2 < \infty$ gives a different realization of $\mathcal{H}^{(s)}$ as a set of sequences $\lambda = \{\lambda_n\}_{n=0}^{\infty}$ with the usual $\ell^2(\mathbb{Z}^+)$ inner product.
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Thus, given a set of moments $\{s_n\}_{n=0}^{\infty}$, we can find real $\{b_n\}_{n=0}^{\infty}$ and positive $\{a_n\}_{n=0}^{\infty}$ so that the moment problem is associated to self-adjoint extensions of the Jacobi matrix,

$$A = \begin{pmatrix} b_0 & a_0 & & & \\ a_1 & b_1 & a_1 & & \\ & a_2 & b_2 & b_3 & \\ & & \ddots & \ddots & \ddots \\ & & & \ddots & \ddots & \ddots \end{pmatrix}.$$
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Consequently, we reveal following correspondences:

<table>
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<tr>
<th>moment sequence</th>
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<td>Orthogonal Polynomials</td>
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<td>three-term recurrence</td>
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Sufficient conditions for determinacy - moment sequence

It is desirable to be able to tell whether the moment problem is determinate (or indeterminate) just by looking at the moment sequence \( \{s_n\}_{n=0}^{\infty} \), or the Jacobi matrix (seq. \( \{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty} \)), or orthogonal polynomials \( \{P_n\}_{n=1}^{\infty} \).
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**Carleman, 1922, 1926**

If

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1) \quad \sum_{n=1}^{\infty} \frac{1}{2^n \sqrt{|s_{2n}|}} = \infty \quad \text{or} \quad 2) \quad \sum_{n=1}^{\infty} \frac{1}{a_n} = \infty
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then the Hamburger moment problem is determinate.

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- Hence, e.g., if \( \{a_n\}_{n=0}^{\infty} \) is bounded or there are \( R, C > 0 \) such that

\[
|s_n| \leq CR^n n!,
\]

for all \( n \) sufficiently large, we have determinate Hamburger m.p. If

\[
|s_n| \leq CR^n (2n)!,
\]

for all \( n \) sufficiently large, we have determinate Stieltjes m.p.
Sufficient conditions for determinacy - Jacobi matrix

Chihara, 1989

Let
\[ \lim_{n \to \infty} b_n = \infty \quad \text{and} \quad \lim_{n \to \infty} \frac{a_n^2}{b_n b_{n+1}} = L < \frac{1}{4}. \]

then the Hamburger moment problem is determinate if

\[ \liminf_{n \to \infty} \sqrt{n} b_n < \frac{1 + \sqrt{1 - 4L}}{1 - \sqrt{1 - 4L}} \]

and indeterminate if the opposite (strict) inequality holds.
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- Chihara uses totally different approach to the problem - concept of chain sequences.
Recall \( \{P_n\}_{n=0}^{\infty} \) are determined by the three-term recurrence

\[
xP_n(x) = a_n P_{n+1}(x) + b_n P_n(x) + a_{n-1} P_{n-1}(x)
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Let us denote by \( \{Q_n\}_{n=0}^\infty \) a polynomial sequence that solve the same recurrence as \( \{P_n\}_{n=0}^\infty \) with initial conditions \( Q_0(x) = 0 \) and \( Q_1(x) = \frac{1}{b_0} \).
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These two polynomial sequences are linearly independent and any solution of the three-term recurrence is a linear combination of them.
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The Hamburger moment problem is determinate if and only if

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- Actually, one can write some \( x \in \mathbb{R} \) instead of zero in the condition.
- It is even necessary and sufficient that there exists a \( z \in \mathbb{C} \setminus \mathbb{R} \) such that both \( \{P_n(z)\}_{n=0}^{\infty} \) and \( \{Q_n(z)\}_{n=0}^{\infty} \) does not belong to \( \ell^2(\mathbb{Z}_+) \).
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### Krein, 1945

Let \( w \) be a density of \( \mu \) (i.e., \( d\mu(x) = w(x)dx \)) where either

1) \( \text{supp}(w) = \mathbb{R} \) and
\[
\int_{\mathbb{R}} \frac{\ln(w(x))}{1 + x^2} dx > -\infty,
\]
or

2) \( \text{supp}(w) = [0, \infty) \) and
\[
\int_0^\infty \frac{\ln(w(x))}{\sqrt{x}(1 + x)} dx > -\infty.
\]

Suppose that for all \( n \in \mathbb{Z}_+ \):
\[
\int_{\mathbb{R}} |x|^n w(x) dx < \infty.
\]

Then the moment problem (Hamburger in case (1), Stieltjes in case(2)) with moments
\[
s_n = \frac{\int x^n w(x) dx}{\int w(x) dx}
\]
is indeterminate.
The problem about describing $\mathcal{M}_H$ was solved by Nevanlinna in 1922 using complex function theory.

A function $\phi$ is called Pick function (beware Herglotz) if it is holomorphic in $\mathbb{C}^+ = \{z \in \mathbb{C} | \Im z > 0\}$ and $\Im \phi(z) \geq 0$ for $z \in \mathbb{C}^+$. Denote the set of Pick functions by $P$. $P \cup \{\infty\}$ denotes the one-point compactification of $P$ ($P$ inherits the topology of holomorphic functions on $\mathbb{C} \setminus \mathbb{R}$).

Nevanlinna, 1922

The solutions of the Hamburger moment problem in the indeterminate case are parametrized via homeomorphism $\phi \mapsto \mu_\phi$ of $P \cup \{\infty\}$ onto $\mathcal{M}_H$ given by

$$\int \mathbb{R} d \mu_\phi(z) x - z = -A(z) \phi(z) - C(z) B(z) \phi(z) - D(z) , \quad z \in \mathbb{C} \setminus \mathbb{R},$$

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The solution $\mu_\phi$ can be then expressed by using Stiltjes-Perron inversion formula.
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František Štampach (FNSPE, CTU)
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- Hence polynomials are not dense in $L^2(d\mu_\vartheta)$. This is a typical situation for solutions of indeterminate moment problems which are not N-extremal.
Nevanlinna functions $A, B, C,$ and $D$

- In some sense, to solve indeterminate Hamburger moment problem means to find the Nevanlinna functions $A, B, C,$ and $D$ (in particular $B$ and $D$).

$A(z) = \sum_{k=0}^{\infty} Q_k(0) Q_k(z)$,  
$C(z) = 1 + \sum_{k=0}^{\infty} P_k(0) Q_k(z)$,  
$B(z) = -1 + \sum_{k=0}^{\infty} Q_k(0) P_k(z)$,  
$D(z) = \sum_{k=0}^{\infty} P_k(0) P_k(z)$, where sums converge locally uniformly in $\mathbb{C}$.

More on $A, B, C, D$: $A, B, C, D$ are entire functions of order $\leq 1$, if the order is 1, the exponential type is 0 [Riesz, 1923]. $A, B, C, D$ have the same order, type and Phragmén-Lindelöf indicator function [Berg and Pedersen, 1994].
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They can be computed by using orthogonal polynomials,

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A(z) = z \sum_{k=0}^{\infty} Q_k(0) Q_k(z), \quad C(z) = 1 + z \sum_{k=0}^{\infty} P_k(0) Q_k(z)
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If $\phi(z) = t \in \mathbb{R} \cup \{\infty\}$ then $\phi \in \mathcal{P} \cup \{\infty\}$ and $\mu_t$ is a discrete measure of the form

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\mu_t = \sum_{x \in \Lambda_t} \rho(x) \delta(x).
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$\Lambda_t$ denotes the set of zeros of $x \mapsto B(x) - D(x)$ (or $x \mapsto B(x)$ if $t = \infty$) and $\sum_{n=0}^{\infty} P_n(x) = B'(x)D(x) - B(x)D'(x)$, $x \in \mathbb{R}$. Measures $\mu_t$, $t \in \mathbb{R} \cup \{\infty\}$, are all N-extremal solutions. They are the only solutions for which polynomials $C[x]$ are dense in $L^2(\mathbb{R}, \mu_t)$ ($\{P_n\}$ forms an orthonormal basis of $L^2(\mathbb{R}, \mu_t)$), [Riesz, 1923]. N-extremal solutions are indeed extreme points in $M_{\mathcal{H}}$ but not the only ones.
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Important solutions 1/2

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If we set

\[ \phi(z) = \begin{cases} 
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for \( \beta \in \mathbb{R} \) and \( \gamma > 0 \), then \( \phi \in \mathcal{P} \) and \( \mu_{\beta, \gamma} \) is absolutely continuous with density

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- The solution \( \mu_{0,1} \) is the one that maximizes certain entropy integral, see Krein’s condition. More general and additional information are provided in [Gabardo, 1992].
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Just restrict oneself to consider only the Pick functions \( \phi \) which have an analytic continuation to \( \mathbb{C} \setminus [0, \infty) \) such that \( \alpha \leq \phi(x) \leq 0 \) for \( x < 0 \), [Pedersen, 1997]
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For the indeterminate Stieljes moment problem there is a slightly more elegant way how to describe \( \mathcal{M}_S \) known as *Krein parametrization*, [Krein, 1967].
Thank you, and see you in Beskydy!