The Moment Problem

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Outline

1. Motivation
2. What the moment problem is?
3. Existence and uniqueness of the solution - operator approach
4. Jacobi matrix and Orthogonal Polynomials
5. Sufficient conditions for determinacy
6. The set of solutions of indeterminate moment problem

Motivation - introduction

- Chebychev's question: *If for some positive function f,*
$$\int_{\mathbb{R}} x^n f(x)dx = \int_{\mathbb{R}} x^n e^{-x^2} dx, \quad n = 0, 1, \ldots$$
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A tough problem: What can be said when there is no longer uniqueness?

Let $I \subset \mathbb{R}$ be an open interval. For a positive measure μ on I the nth moment is defined as

$$\int_I x^n d\mu(x), \qquad \text{(provided the integral exists).}$$

Suppose a real sequence $\{s_n\}_{n\geq 0}$ is given. The moment problem on I consists of solving the following three problems:

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   If so,

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One can restrict oneself to cases:

- $I = \mathbb{R}$ - *Hamburger* moment problem ($\mathcal{M}_H$ = set of solutions)
- $I = [0, +\infty)$ - *Stieltjes* moment problem ($\mathcal{M}_S$ = set of solutions)
- $I = [0, 1]$ - *Hausdorff* moment problem

Hausdorff, 1923

The moment problem has a solution on $[0, 1]$ iff sequence $\{s_n\}_{n\geq 0}$ is *completely monotonic*, i.e.,

$$(-1)^k(\Delta^k s)_n \geq 0$$

for all $k, n \in \mathbb{Z}_+$, where $(\Delta s)_n = s_{n+1} - s_n$.

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Consequently, we will further discuss the Stieltjes and Hamburger moment problem only.

Existence of the solution

- For $\{s_n\}_{n\geq 0}$, we denote $H_N(s)$ the $N \times N$ Hankel matrix with entries $(H_N(s))_{ij} := s_{i+j}$, $i, j \in \{0, 1, \dots N-1\}$.

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- Define two sesquilinear forms H_N and S_N on $\mathbb{C}^N$ by

$$H_N(x,y) := \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \overline{x_i} y_j s_{i+j} \quad \text{and} \quad S_N(x,y) := \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \overline{x_i} y_j s_{i+j+1}.$$

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- Let $\mu \in \mathcal{M}_H$ or $\mu \in \mathcal{M}_S$ with infinite support. By observing that

$$H_N(y,y) = \int \left| \sum_{i=0}^{N-1} y_i x^i \right|^2 d\mu(x) \quad \text{and} \quad S_N(y,y) = \int x \left| \sum_{i=0}^{N-1} y_i x^i \right|^2 d\mu(x),$$

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Necessary condition for the existence

A necessary condition for the Hamburger moment problem to have a solution (with infinite support) is the sesquilinear form H_N is PD for all $N \in \mathbb{Z}_+$. A necessary condition for the Stieltjes moment problem to have a solution (with infinite support) is both sesquilinear forms H_N and S_N are PD for all $N \in \mathbb{Z}_+$.

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- For $P, Q \in \mathbb{C}[x]$,
$$P(x) = \sum_{k=0}^{N-1} a_k x^k, \quad \text{and} \quad Q(x) = \sum_{k=0}^{N-1} b_k x^k,$$
  define positive definite inner product
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- Especially,

$$\langle 1, A^n 1 \rangle = s_n, \quad n \in \mathbb{N}.$$

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- Especially, for $f(x) = x^n$, one finds

$$s_n = \langle 1, A^n 1 \rangle = \langle 1, (A')^n 1 \rangle = \int_{\mathbb{R}} x^n d\mu(x),$$

since $\mathrm{Dom}(A^n) \subset \mathrm{Dom}((A')^n)$.

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Theorem (Existence)

i) A necessary and sufficient condition for $\mathcal{M}_H \neq \emptyset$ (with infinite support) is

$$\det H_N(s) > 0 \quad \text{for all } N \in \mathbb{N}.$$

ii) A necessary and sufficient condition for $\mathcal{M}_S \neq \emptyset$ (with infinite support) is

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- Historically, this result has not been obtained by using the spectral theorem that was invented later.

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Theorem (Uniqueness)

i) A necessary and sufficient condition for the Hamburger moment problem to be determinate is that the operator A is essentially self-adjoint (i.e., it has a unique self-adjoint extension).

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- The other direction is even less clear. For not only is it not obvious, it is **false** that every solution of the moment problem arise from some measure given by spectral measure of some self-adjoint extension.
- A solution of the moment problem which comes from a self-adjoint extension of A is called *N-extremal* solution (von Neumann [Simon], extremal [Shohat-Tamarkin]).

- Consider set $\{1, x, x^2, \dots\} \subset \mathcal{H}^{(s)}$ which is linearly independent (H_N PD) and span $\mathcal{H}^{(s)}$.

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- By applying the Gramm-Schmidt procedure, we obtain an orthonormal basis $\{P_n\}_{n=0}^{\infty}$ for $\mathcal{H}^{(s)}$.
- By construction, P_n is a polynomial of degree n with real coefficients and

$$\langle P_m, P_n \rangle = \delta_{mn}$$

for all $m, n \in \mathbb{Z}_+$. These are well-known *Orthogonal Polynomials*.

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- By applying the Gramm-Schmidt procedure, we obtain an orthonormal basis $\{P_n\}_{n=0}^{\infty}$ for $\mathcal{H}^{(s)}$.
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$$\langle P_m, P_n \rangle = \delta_{mn}$$
for all $m, n \in \mathbb{Z}_+$. These are well-known *Orthogonal Polynomials*.
- $\{P_n\}_{n=0}^{\infty}$ are determined by moment sequence $\{s_n\}_{s=0}^{\infty}$,

$$P_n(x) = \frac{1}{\sqrt{\det[H_{n+1}(s)H_n(s)]}} \begin{vmatrix} s_0 & s_1 & \dots & s_n \\ s_1 & s_2 & \dots & s_{n+1} \\ \vdots & \vdots & & \vdots \\ s_{n-1} & s_n & \dots & s_{2n-1} \\ 1 & x & \dots & x^n \end{vmatrix}.$$

- Since $\mathrm{span}(1, x, \dots, x^n) = \mathrm{span}(P_0, P_1, \dots, P_n)$, $xP_n(x)$ has an expansion in $P_0, P_1, \dots, P_{n+1}$.

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  where $a_n > 0$ and $b_n \in \mathbb{R}$.
- Hence A has, in the basis $\{P_n\}_{n=0}^{\infty}$, has tridiagonal matrix representation and $\operatorname{Dom}(A)$ is the set of sequences of finite support.

- The realization of elements of $\mathcal{H}^{(s)}$ as $\sum_{n=0}^{\infty} \lambda_n P_n$, with $\sum_{n=0}^{\infty} |\lambda_n|^2 < \infty$ gives a different realization of $\mathcal{H}^{(s)}$ as a set of sequences $\lambda = \{\lambda_n\}_{n=0}^{\infty}$ with the usual $\ell^2(\mathbb{Z}_+)$ inner product.

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- Consequently, we reveal following correspondences:

$$\begin{array}{ccc} \text{moment sequence} & \leftrightarrow & \text{Jacobi matrix} \\ \updownarrow & & \updownarrow \\ \text{Orthogonal Polynomials} & \leftrightarrow & \text{three-term recurrence} \end{array}$$

Sufficient conditions for determinacy - moment sequence

It is desirable to be able to tell whether the moment problem is determinate (or indeterminate) just by looking at the moment sequence $\{s_n\}_{n=0}^{\infty}$, or the Jacobi matrix (seq. $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$), or orthogonal polynomials $\{P_n\}_{n=1}^{\infty}$.

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Carleman, 1922, 1926

If

$$1)\ \sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{|s_{2n}|}} = \infty \qquad \text{or} \qquad 2)\ \sum_{n=1}^{\infty} \frac{1}{a_n} = \infty$$

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- Hence, e.g., if $\{a_n\}_{n=0}^{\infty}$ is bounded or there are $R, C > 0$ such that

$$|s_n| \leq CR^n n!,$$

  for all n sufficiently large, we have determinate Hamburger m.p. If

$$|s_n| \leq CR^n (2n)!,$$

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Chihara, 1989

Let

$$\lim_{n\to\infty} b_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{a_n^2}{b_n b_{n+1}} = L < \frac{1}{4}.$$

then the Hamburger moment problem is determinate if

$$\liminf_{n\to\infty} \sqrt[n]{b_n} < \frac{1+\sqrt{1-4L}}{1-\sqrt{1-4L}}$$

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- Chihara uses totally different approach to the problem - concept of chain sequences.

- Recall $\{P_n\}_{n=0}^{\infty}$ are determined by the three-term recurrence

$$xP_n(x) = a_nP_{n+1}(x) + b_nP_n(x) + a_{n-1}P_{n-1}(x)$$

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- Let us denote by $\{Q_n\}_{n=0}^{\infty}$ a polynomial sequence that solve the same recurrence as $\{P_n\}_{n=0}^{\infty}$ with initial conditions $Q_0(x) = 0$ and $Q_1(x) = \frac{1}{b_0}$.

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The Hamburger moment problem is determinate if and only if

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- It is even necessary and sufficient that there exists a $z \in \mathbb{C} \setminus \mathbb{R}$ such that both $\{P_n(z)\}_{n=0}^{\infty}$ and $\{Q_n(z)\}_{n=0}^{\infty}$ does not belong to $\ell^2(\mathbb{Z}_+)$.

Sufficient conditions for indeterminacy - density of measure

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Krein, 1945

Let w be a density of μ (i.e., $d\mu(x) = w(x)dx$) where either
1) $\operatorname{supp}(w) = \mathbb{R}$ and

$$\int_{\mathbb{R}} \frac{\ln(w(x))}{1+x^2}dx > -\infty,$$

or
2) $\operatorname{supp}(w) = [0,\infty)$ and

$$\int_0^{\infty} \frac{\ln(w(x))}{\sqrt{x}(1+x)}dx > -\infty.$$

Suppose that for all $n \in \mathbb{Z}_+$:

$$\int_{\mathbb{R}} |x|^n w(x)dx < \infty.$$

Then the moment problem (Hamburger in case (1), Stieltjes in case(2)) with moments

$$s_n = \frac{\int x^n w(x)dx}{\int w(x)dx}$$

is **in**determinate.

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The solutions of the Hamburger moment problem in the indeterminate case are parametrized via homeomorphism $\phi \mapsto \mu_\phi$ of $\mathcal{P} \cup \{\infty\}$ onto $\mathcal{M}_H$ given by

$$\int_{\mathbb{R}} \frac{d\mu_\phi(x)}{x - z} = -\frac{A(z)\phi(z) - C(z)}{B(z)\phi(z) - D(z)}, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

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- The solution μ_ϕ can be then expressed by using Stiltjes-Perron inversion formula.

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- Extreme points are dense in $\mathcal{M}_H$.

Example due to Stieltjes

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- Hence polynomials are not dense in $L^2(d\mu_\vartheta)$. This is a typical situation for solutions of indeterminate moment problems which are not N-extremal.

Nevanlinna functions A,B,C, and D

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More on A,B,C,D:

- A,B,C,D are entire functions of order ≤ 1, if the order is 1, the exponential type is 0 [Riesz, 1923]
- A,B,C,D have the same order, type and Phragmén-Lindenlöf indicator function [Berg and Pedersen, 1994]

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- N-extremal solutions are indeed extreme points in $\mathcal{M}_H$ - but not the only ones.

Important solutions 2/2

- If we set

$$\phi(z) = \begin{cases} \beta + i\gamma, & \Im z > 0, \\ \beta - i\gamma, & \Im z < 0, \end{cases}$$

  for $\beta \in \mathbb{R}$ and $\gamma > 0$, then $\phi \in \mathcal{P}$ and $\mu_{\beta,\gamma}$ is absolutely continuous with density

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- Polynomials $\mathbb{C}[x]$ are not dense in $L^1(\mathbb{R}, \mu_{\beta,\gamma})$.
- The solution $\mu_{0,1}$ is the one that maximizes certain entropy integral, see Krein's condition. More general and additional information are provided in [Gabardo, 1992].

Nevanlinna parametrization in the case of Stieltjes moment problem

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- For the indeterminate Stieljes moment problem there is a sligtly more elegant way how to describe $\mathcal{M}_S$ known as *Krein parametrization*, [Krein, 1967].

Thank you, and see you in Beskydy!