Spectral analysis of certain doubly infinite Jacobi matrices via characteristic function

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Operator Theory 26

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Contents

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- To the doubly infinite Jacobi matrix

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where $\lambda_n \in \mathbb{C}$ and $w_n \in \mathbb{C} \setminus \{0\}$, we associate two operators $J_{\min}$ and $J_{\max}$ acting on $\ell^2(\mathbb{Z})$.

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- Both operators $J_{\min}$ and $J_{\max}$ are closed and densely defined. They are related as

$$J_{\max}^* = C J_{\min} C \quad \text{and} \quad J_{\min}^* = C J_{\max} C$$

  where C is the complex conjugation operator, $(Cx)_n = \overline{x_n}$.

Proper case and spectrum of Jacobi operator

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  where the essential spectrum has the simple characterization:

$$\sigma_{ess}(J) = \{z \in \mathbb{C} \mid \mathrm{Ran}(J - z) \text{ is not closed } \}.$$

Contents

Function $\mathfrak{F}$

Definition:

For $\{x_n\}_{n=N_1}^{N_2}$, $N_1, N_2 \in \mathbb{Z} \cup \{-\infty, +\infty\}$, $N_1 \leq N_2$, such that

$$\sum_{k=N_1}^{N_2-1} |x_k x_{k+1}| < \infty,$$

we define

$$\mathfrak{F}\left(\{x_k\}_{k=N_1}^{N_2}\right) := 1 + \sum_{m=1}^{\infty} (-1)^m \sum_{k \in \mathcal{I}(N_1,N_2,m)} \prod_{j=1}^{m} x_{k_j} x_{k_j+1}$$

where

$$\mathcal{I}(N_1, N_2, m) = \{k \in \mathbb{Z}^m \mid k_j + 2 \leq k_{j+1} \text{ for } 1 \leq j \leq m-1,\ N_1 \leq k_1,\ k_m < N_2\}.$$

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Function $\mathfrak{F}$ is well defined, we have the estimate

$$\left|\mathfrak{F}\left(\{x_k\}_{k=N_1}^{N_2}\right)\right| \leq \exp\left(\sum_{k=N_1}^{N_2-1} |x_k x_{k+1}|\right).$$

Properties of $\mathfrak{F}$

- For example, if $N_1 = -\infty$ and $N_2 = \infty$, one has

$$
\begin{aligned}
&\mathfrak{F}\left(\{x_k\}_{k=-\infty}^{\infty}\right) \\
&= 1 + \sum_{m=1}^{\infty}(-1)^m \sum_{k_1=-\infty}^{\infty} \sum_{k_2=k_1+2}^{\infty} \cdots \sum_{k_m=k_{m-1}+2}^{\infty} x_{k_1}x_{k_1+1}x_{k_2}x_{k_2+1}\ldots x_{k_m}x_{k_m+1} \\
&= 1 - (\dots + x_1x_2 + x_2x_3 + x_3x_4 + \dots) \\
&\quad + (\dots + x_1x_2x_3x_4 + x_1x_2x_4x_5 + \dots + x_2x_3x_4x_5 + x_2x_3x_5x_6 + \dots +) \\
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- Function $\mathfrak{F}$ has interesting properties and satisfies many algebraic and combinatorial identities.
- The relation between $\mathfrak{F}$ and **tridiagonal matrices** may be indicated by the identity

$$\mathfrak{F}(\{x_k\}_{k=1}^{n}) = \det \begin{pmatrix} 1 & x_1 & & & \\ x_2 & 1 & x_2 & & \\ & \ddots & \ddots & \ddots & \\ & & x_{n-1} & 1 & x_{n-1} \\ & & & x_n & 1 \end{pmatrix}.$$

Properties of $\mathfrak{F}$ - cont.

- Function $\mathfrak{F}$ is also closely related with continued fractions:

$$\frac{\mathfrak{F}\left(\{x_n\}_{n=2}^{\infty}\right)}{\mathfrak{F}\left(\{x_n\}_{n=1}^{\infty}\right)} = \cfrac{1}{1 - \cfrac{x_1 x_2}{1 - \cfrac{x_2 x_3}{1 - \cfrac{x_3 x_4}{1 - \dots}}}}.$$

where the RHS converges (as the sequence of corresponding truncations) whenever

$$\sum_{n=1}^{\infty} |x_n x_{n+1}| < \infty \quad \text{and} \quad \mathfrak{F}\left(\{x_n\}_{n=1}^{\infty}\right) \neq 0.$$

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If you are interested:

1. F. Š. and P. Šťovíček, *Linear Alg. Appl.* (2011), **arXiv:1011.1241**.
2. F. Š. and P. Šťovíček: *Linear Alg. Appl.* (2013), **arXiv:1201.1743**.
3. F. Š. and P. Šťovíček: *J. Math. Anal. Appl.* (2014), **arXiv:1301.2125**.
4. F. Š. and P. Šťovíček: *Linear Alg. Appl.* (2015), **arXiv:1403.8083**.

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- We use the following notation:

$$\mathbb{C}_0^{\lambda} := \mathbb{C} \setminus \overline{\{\lambda_n \mid n \in \mathbb{Z}\}},$$

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- This assumption determines the class of matrices $\mathcal{J}$ form which we can define the **characteristic function**:

$$F_{\mathcal{J}}(z) := \mathfrak{F}\left(\left\{ \frac{\gamma_n^2}{z - \lambda_n} \right\}_{n=-\infty}^{\infty} \right), \quad \forall z \in \mathbb{C}_0^\lambda,$$

where $\{\gamma_n\}$ is any sequence satisfying the difference equation $\gamma_n \gamma_{n+1} = w_n$, $\forall n \in \mathbb{Z}$.

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  where $\{\gamma_n\}$ is any sequence satisfying the difference equation $\gamma_n \gamma_{n+1} = w_n$, $\forall n \in \mathbb{Z}$.
- Function $F_{\mathcal{J}}$ is well define and entire on $\mathbb{C}_0^{\lambda}$. Further, $F_{\mathcal{J}}$ meromorphic on $\mathbb{C} \setminus \mathrm{der}(\lambda)$ having poles of finite order (or removable singularities) at points $z = \lambda_n$.

The zero set of characteristic function

- It is not clear whether the assumption

$$\sum_{n=-\infty}^{\infty} \left| \frac{w_n^2}{(\lambda_n - z)(\lambda_{n+1} - z)} \right| < \infty, \text{ for one } z \in \mathbb{C}_0^{\lambda}$$

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- Assuming, additionally, that $F_{\mathcal{J}} \neq 0$ identically on $\mathbb{C}_0^{\lambda}$, then $J_{\min} = J_{\max}$. This assumption holds if, for example, $\{\lambda_n\}$ is located in a sector in $\mathbb{C}$ with an angle $< 2\pi$.

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Theorem:

Under the above mentioned assumptions, one has

$$\sigma(J) \cap \mathbb{C}_0^{\lambda} = \sigma_p(J) \cap \mathbb{C}_0^{\lambda} = \{z \in \mathbb{C}_0^{\lambda} \mid F_{\mathcal{J}}(z) = 0\}.$$

On eigenvectors and multiplicities

- For $z \in \mathbb{C}_0^{\lambda}$, put

$$f_n(z) := \left(\prod_{k=1}^{n}{}^{*} \frac{w_{k-1}}{z - \lambda_k} \right) \mathfrak{F}\left(\left\{ \frac{\gamma_k^2}{\lambda_k - z} \right\}_{k=n+1}^{\infty} \right)$$

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Theorem:

i) All the eigenvalues of J have geometric multiplicity equal to one with $f(z)$ the eigenvector corresponding to the eigenvalue $z \in \mathbb{C}_0^\lambda$.
ii) Suppose, in addition, that $\mathbb{C} \setminus \operatorname{der}(\lambda)$ is connected. Then $\sigma_p(J)$ has no accumulation point in $\mathbb{C} \setminus \operatorname{der}(\lambda)$ and the algebraic multiplicity of an eigenvalue $z \in \mathbb{C}_0^\lambda$ of J coincides with the order of z as a root of $F_{\mathcal{J}}$. In this case, the space of generalized eigenvectors is spanned by

$$f(z), f'(z), \dots, f^{(m-1)}(z)$$

where m is the algebraic multiplicity of z.

The Green function

- Under the two main assumptions, the resolvent set $\rho(J) \neq \emptyset$ and the Green function

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is, for $z \in \rho(J) \setminus \operatorname{der}(\lambda)$, given by the formula

$$G_{i,j}(z) = -\frac{1}{w_{\max(i,j)}} \left(\prod_{k=\min(i,j)}^{\max(i,j)} \frac{w_k}{z-\lambda_k} \right) \frac{\mathcal{F}_{-\infty}^{\min(i,j)-1}(z)\, \mathcal{F}_{\max(i,j)+1}^{\infty}(z)}{\mathcal{F}_{-\infty}^{\infty}(z)}$$

where we denote

$$\mathcal{F}_m^n(z) := \mathfrak{F}\left(\left\{ \frac{\gamma_k^2}{\lambda_k - z} \right\}_{k=m}^{n} \right), \quad m, n \in \mathbb{Z} \cup \{\pm\infty\}.$$

A summation formula for eigenvectors

Proposition:

If $F_{\mathcal{J}}(z) = 0$, for some $z \in \mathbb{C}_0^{\lambda}$, then

$$\sum_{n=-\infty}^{\infty} f_n^2(z) = A(z) F'_{\mathcal{J}}(z)$$

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Corollary

If there exists and eigenvector $v(z)$ to an eigenvalue $z \in \mathbb{C}_0^{\lambda}$ of J such that

$$\sum_{n=-\infty}^{\infty} v_n^2(z) = 0.$$

Then J is not diagonalizable.

Local regularization - from $\mathbb{C}_0^\lambda$ to $\mathbb{C} \setminus \operatorname{der}(\lambda)$

- All the spectral results have been restricted to the set $\mathbb{C}_0^\lambda$. Recall, for example,

$$\sigma(J) \cap \mathbb{C}_0^\lambda = \sigma_p(J) \cap \mathbb{C}_0^\lambda = \{z \in \mathbb{C}_0^\lambda \mid F_{\mathcal{J}}(z) = 0\}.$$

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- By using certain procedure (a local regularization), we can extend the results from $\mathbb{C}_0^\lambda$ to $\mathbb{C} \setminus \operatorname{der}(\lambda)$, getting, for example,

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- For the sake of brevity, we do not explain the details of this local regularization procedure in this talk. Rather we describe a global regularization of the characteristic function in 3 different cases and provide illustrating examples ...

Contents

I. Compact case - regularization

- If we assume
$$\lambda \in \ell^p(\mathbb{Z}) \text{ for some } p \in \mathbb{N} \quad \text{and} \quad w \in \ell^2(\mathbb{Z}),$$
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- Then one can get rid of all the nonzero singularities of $F_{\mathcal{J}}$ and f by putting

$$\tilde{F}_{\mathcal{J}}(z) := \Phi_p(z) F_{\mathcal{J}}(z) \quad \text{and} \quad \tilde{f}(z) := \Phi_p^+(z) f(z).$$

  Functions $\tilde{F}_{\mathcal{J}}$ and $\tilde{f}$ are entire on $\mathbb{C} \setminus \{0\}$.

I. Compact case - spectral results

Proposition:

If

$$w \in \ell^2(\mathbb{Z}) \quad \text{and} \quad \lambda \in \ell^p(\mathbb{Z}) \quad \text{for some} \quad p \geq 1,$$

then

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Remark: In this case for $p \geq 2$, one can show that J is a compact operator from the Schatten–von Neumann class $\mathcal{S}_p$ and

$$\tilde{F}_{\mathcal{J}}(1/z) = \det_p(1 - zJ), \quad \forall z \in \mathbb{C}.$$

So the above statement may be deduced from results of the theory of regularized determinants.

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- For $\alpha \in \mathbb{C} \setminus \mathbb{Z}$ and $\beta \in \mathbb{C} \setminus \{0\}$ put

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- Note that $\sigma(J)$ does not depend on β. Hence, for any $\alpha \in \mathbb{R} \setminus \mathbb{Z}$, J is a non-self-adjoint operator with purely real spectrum.

I. Compact resolvent case - regularization

- If we assume $\lambda_n \neq 0$, $\forall n \in \mathbb{Z}$,

$$\sum_{n=-\infty}^{\infty} \left| \frac{w_n^2}{\lambda_n \lambda_{n+1}} \right| < \infty \quad \text{and} \quad \sum_{n=-\infty}^{\infty} \frac{1}{|\lambda_n|^p} < \infty, \quad \text{for some } p \in \mathbb{N},$$

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Functions $\tilde{F}_{\mathcal{J}}$ and $\tilde{f}$ are entire.

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Let $\lambda_n \neq 0$ and $w_n \neq 0$ be such that

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Further, let $\tilde{F}_{\mathcal{J}}$ does not vanish identically on $\mathbb{C}$. Then $J_{\min} = J_{\max} =: J$,

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Remark: For $z \in \mathbb{C}$, we may introduce the operator $A(z)$ determined by equalities

$$A(z)e_n = \frac{w_{n-1}}{\sqrt{\lambda_{n-1}\lambda_n}} e_{n-1} - \frac{z}{\lambda_n} e_n + \frac{w_n}{\sqrt{\lambda_n \lambda_{n+1}}} e_{n+1}, \quad n \in \mathbb{Z},$$

which is, if $p \geq 2$, in $\mathcal{S}_p$. In addition, it can be shown that

$$z \in \sigma(J) \iff -1 \in \sigma((A(z))$$

and we have

$$\det_p(1 + A(z)) = \tilde{F}_{\mathcal{J}}(z), \quad z \in \mathbb{C}.$$

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are to be used to regularize $F_{\mathcal{J}}$ and f:

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- For $q, \beta \in \mathbb{C}$, $0 < |q| < 1$ and $\beta \neq 0$, put

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$$\tilde{f}_n(z) = z^{-n} \beta^n q^{n(n-1)/4} \, {}_0\tilde{\phi}_1\left(-; z^{-1} q^{n+1}; q, -q^{n+1} z^{-2} \beta^2\right).$$

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$$\lambda_n = q^n \quad \text{and} \quad w_n = \beta q^{n/2}, \quad \text{for } n \in \mathbb{Z}.$$

- For $z \neq 0$, the regularized characteristic function equals

$$\tilde{F}_{\mathcal{J}}(z) = \left(z, qz^{-1}, -\beta^2 z^{-1}; q\right)_\infty = \prod_{k=0}^{\infty} \left(1 - zq^k\right)\left(1 - \frac{q^k}{z}\right)\left(1 + \frac{\beta^2 q^k}{z}\right).$$

- One has,

$$\sigma(J) \setminus \{0\} = \sigma_p(J) = q^{\mathbb{Z}} \cup (-\beta^2) q^{\mathbb{N}_0}.$$

  Here, indeed, $\sigma_{ess}(J) = \sigma_c(J) = \{0\}$.

- The n-th element of the eigenvector to the eigenvalue $z \in \sigma_p(J)$ reads

$$\tilde{f}_n(z) = z^{-n} \beta^n q^{n(n-1)/4} \, {}_0\tilde{\phi}_1 \left(-; z^{-1} q^{n+1}; q, -q^{n+1} z^{-2} \beta^2\right).$$

- Whenever $q \in (-1, 1)$ and β is purely imaginary, then J is a non-self-adjoint operator with purely real spectrum. In this case, J is diagonalizable if and only if $\beta \notin \mathrm{i}q^{\mathbb{Z}/2}$.

Thank you!