Nevanlinna functions and orthognality relations for q-Lommel polynomials

František Štampach

Faculty of Nuclear Sciences and Physical Engineering, CTU in Prague

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Contents

- By q-Lommel polynomials $h_{n,\nu}(w;q)$ we mean those functions arising in the relation

$$J_{\nu+n}(w;q) = h_{n,\nu}(w^{-1};q)J_{\nu}(w;q) - h_{n-1,\nu+1}(w^{-1};q)J_{\nu-1}(w;q)$$

where $J_{\nu}(w;q)$ denotes the Hahn-Exton q-Bessel function,

$$J_{\nu}(w;q) = w^{\nu}\frac{(q^{\nu+1};q)_{\infty}}{(q;q)_{\infty}}\,{}_1\phi_1\left(0;q^{\nu+1};q,qw^2\right).$$

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- Function $h_{n,\nu}(w;q)$ are Laurent polynomials in w and polynomials in q^ν and are generated by recurrence

$$h_{n-1,\nu}(w;q) - (w^{-1} + w(1-q^\nu))h_{n,\nu}(w;q) + h_{n+1,\nu}(w;q) = 0,$$

with initial conditions $h_{-1,\nu}(w;q) = 0$ and $h_{0,\nu}(w;q) = 1$.

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- q-Lommel polynomials have been intensively studied in 90's by Koelink, Van Aschee, Swarttouw, and others.

Monic q-Lommel polynomials

- The monic version of q-Lommel polynomials $F_n(w; q, x)$ are generated by recurrence

$$u_{n+1} = (x - (w^{-2} + 1)q^{-n})u_n - w^{-2}q^{-2n+1}u_{n-1}$$

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- Polynomials $F_n(w;q,x)$ are related with $h_{n,\nu}(w;q)$ by identity

$$h_{n,\nu}(w;q) = (-1)^n w^n q^{\frac{1}{2}n(n-1)} F_n(w;q,q^\nu).$$

Notice we identify $x = q^\nu$.

Proposition

The Hamburger as well as the Stieltjes moment problem associated with polynomials $F_n(w; q, x)$ is indeterminate if and only if $q < w^{-2} < 1/q$.

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Proof.

Based on explicit formula for corresponding orthonormal polynomials $P_n(w; q, 0)$ and $Q_n(w; q, 0)$ from which one deduces both are square summable iff $q < w^{-2} < 1/q$.
The indeterminacy of the Stieltjes moment problem then follows from the fact that

$$\lim_{n\to\infty} \frac{P_n(w; q, 0)}{Q_n(w; q, 0)} < 0,$$

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Orthogonality relation [Koelink]

For $m, n \in \mathbb{Z}_+$, it holds

$$\sum_{k=1}^{\infty} \frac{{}_1\phi_1(0; qw^{-2}; q, q\xi_k)}{\partial_x|_{x=\xi_k}\, {}_1\phi_1(0; qw^{-2}; q, x)} F_n(w; q, \xi_k) F_m(w; q, \xi_k) = -w^{-2n} q^{-n^2} \delta_{mn}.$$

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Proposition

For $|t| < \min(1, w^2)$, it holds

$$\sum_{n=0}^{\infty} q^{\binom{n}{2}} F_n(w; q, x)(-t)^n = \frac{1}{(1-t)(1-w^{-2}t)}\, {}_2\phi_2(0, q; qt, qw^{-2}t; q, xt).$$

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By denoting the LHS of the above formula $V(t)$, one finds V fulfills the q-difference equation

$$(1-t)(1-w^{-2}t)V(t) = 1 - xtV(qt)$$

which leads to the result by iterating. □

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The standard use of the Darboux's method provides us with the following limit relations:

$$\lim_{n\to\infty} (-1)^n q^{\binom{n}{2}} F_n(w; q, x) = \frac{1}{1-w^{-2}}\, {}_1\phi_1(0; w^{-2}q; q, x), \quad \text{if } w > 1,$$

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and

$$\lim_{n\to\infty} \frac{(-1)^n}{n} q^{\binom{n}{2}} F_n(1; q, x) = {}_1\phi_1(0; q; q, x), \quad \text{for } w = 1.$$

Nevanlinna parametrization – generalities

- Recall Nevanlinna functions A, B, C, and D defined by

$$A(z)=z\sum_{n=0}^{\infty}Q_n(0)Q_n(z),\qquad B(z)=-1+z\sum_{n=0}^{\infty}Q_n(0)P_n(z),$$
$$C(z)=1+z\sum_{n=0}^{\infty}P_n(0)Q_n(z),\qquad D(z)=z\sum_{n=0}^{\infty}P_n(0)P_n(z),$$

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  where P_n and Q_n are orthonormal polynomials of the first and second kind, respectively, are of the greatest interest for the indeterminate Hamburger moment problem.

- By the Nevanlinna theorem, all measures of orthogonality μ_φ for which

$$\int_{\mathbb{R}} P_n(x)P_m(x)d\mu_\varphi(x) = \delta_{mn}, \quad m, n \in \mathbb{Z}_+,$$

  are parametrized according to

$$\int_{\mathbb{R}} \frac{d\mu_\varphi(x)}{z-x} = \frac{A(z)\varphi(z) - C(z)}{B(z)\varphi(z) - D(z)}, \quad z \in \mathbb{C}\setminus\mathbb{R},$$

  where $\varphi \in \mathcal{P} \cup \{\infty\}$ and $\mathcal{P}$ is the space of Pick functions.

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$$A(w; q, z) = \frac{zq}{1 - w^{-2}} \sum_{n=1}^{\infty} (-1)^{n+1} (w^{2n} - 1) q^{\binom{n}{2}} F_{n-1}(w; q, z)$$

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$$= \frac{zq}{1-w^{-2}} \left[\underbrace{w^2 \sum_{n=0}^{\infty} q^{\binom{n}{2}} F_n(w;q,qx)(-qw^2)^n}_{\textbf{gerating function} \text{ formula with } t=qw^2} - \underbrace{\sum_{n=0}^{\infty} q^{\binom{n}{2}} F_n(w;q,qx)(-q)^n}_{\text{...and similarly with } t=q} \right]$$

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- By this way (and using simple identity for q-hypergeometric series) one arrives at the formula

$$A(w;q,z) = \frac{1}{1-w^{-2}} \left[{}_1\phi_1(0; w^{-2}q; q, qz) - {}_1\phi_1(0; w^2q; q, w^2qz) \right].$$

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- Similar computation leads to formulas for B, C, and D, and the result is . . .

Theorem

Let $1 \neq w^{-2} \in (q, q^{-1})$ *then the entire functions from the Nevanlinna parametrization are as follows:*

$$A(w; q, z) = \frac{w^2}{w^2 - 1}\left[{}_1\phi_1(0; w^{-2}q; q, qz) - {}_1\phi_1(0; w^2q; q, w^2qz)\right],$$

$$B(w; q, z) = \frac{1}{1 - w^2}\left[w^2\,{}_1\phi_1(0; w^{-2}q; q, z) - {}_1\phi_1(0; w^2q; q, zw^2)\right],$$

$$C(w; q, z) = \frac{1}{1 - w^2}\left[{}_1\phi_1(0; w^{-2}q; q, qz) - w^2\,{}_1\phi_1(0; w^2q; q, w^2qz)\right],$$

$$D(w; q, z) = \frac{1}{w^2 - 1}\left[{}_1\phi_1(0; w^{-2}q; q, z) - {}_1\phi_1(0; w^2q; q, zw^2)\right].$$

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If $w = 1$ then we have

$$A(1;q,z) = -z\frac{\partial}{\partial z}\,{}_1\phi_1(0;q;q,qz), \quad B(1;q,z) = z^2\frac{\partial}{\partial z}\left[z^{-1}\,{}_1\phi_1(0;q;q,z)\right],$$

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  for the Pick function φ in the Nevanlinna parametrization of the Stieltjes transform of μ_t.
- Measures μ_t are purely discrete with unbounded support. Moreover,

$$\operatorname{supp} \mu_t \subset [0, \infty) \quad \text{iff} \quad t \in [\alpha, 0] \quad \text{where} \quad \alpha = \begin{cases} -1, & \text{if } w \geq 1, \\ -w^{-2}, & \text{if } w < 1. \end{cases}$$

- For a simple form of the following expressions we use notation

$$\phi_w(z) := {}_1\phi_1(0; w^{-2}q; q, z), \quad \text{and} \quad \psi_w(z) := {}_1\phi_1(0; w^2q; q, zw^2).$$

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Let $1 \neq w^{-2} \in (q, q^{-1})$ then all N-extremal measures $\mu_t = \mu_t(w; q)$ are of the form

$$\mu_t = \sum_{x \in \mathfrak{Z}_t} \frac{w^2 - 1}{\phi_w(x)\psi_w'(x) - \psi_w(x)\phi_w'(x)}\, \varepsilon_x$$

where

$$\mathfrak{Z}_t = \mathfrak{Z}_t(w; q) = \{x \in \mathbb{R} \mid (t+1)\psi_w(x) = (w^2 t + 1)\phi_w(x)\}$$

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- By using identity (which is $AD - BC = 1$)

$$w^2\phi_w(z)\psi_w(qz) - \phi_w(qz)\psi_w(z) = w^2 - 1, \quad w \neq 1,$$

one finds the measure derived by Koelink is μ_{-1}, and the orthogonality relation reads

$$\sum_{k=1}^{\infty} \frac{\phi_w(q\xi_k)}{\phi_w'(\xi_k)} F_n(w; q, \xi_k) F_m(w; q, \xi_k) = -w^{-2n} q^{-n^2} \delta_{mn}$$

where $\{\xi_k \mid k \in \mathbb{N}\}$ are all zeros of the function ϕ_w.

Another measure of orthogonality

- Similar orthogonality relation with N-extremal measure $\mu_{-w^{-2}}$ reads

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  where $\{\eta_k \mid k \in \mathbb{N}\}$ are all zeros of the function ψ_w.
- Both measures μ_{-1} and $\mu_{-w^{-2}}$ are supported in $(0,\infty)$ and both correspond to the spectral measure of the Friedrichs extension of associated Jacobi matrix:

$$\mu_{-1} \text{ if } q < w^{-2} < 1, \quad \text{and} \quad \mu_{-w^{-2}} \text{ if } 1 < w^{-2} < 1/q.$$

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Recall $\phi_1(z) = {}_1\phi_1(0; q; q, z)$.

Proposition

For $w = 1$, all N-extremal measures $\mu_t = \mu_t(1; q)$ are of the form

$$\mu_t = -\sum_{x \in \mathfrak{Y}_t} \frac{1}{\phi_1'(x) + x\phi_1''(x)} \varepsilon_x$$

where

$$\mathfrak{Y}_t = \mathfrak{Y}_t(q) = \{x \in \mathbb{R} \mid x(t+1)\phi_1'(x) = t\phi_1(x)\}$$

and ε_x stands for the Dirac measure supported on $\{x\}$.

Absolutely continuous measures of orthogonality

- An example of two-parametric family of absolutely continuous measures $\mu_{\beta,\gamma}$ of orthogonality corresponds to the choice of the Pick function φ as

$$\varphi(z) := \beta + i\gamma \operatorname{sgn} \Im z, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

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- By setting $\beta = -1$ (for simplicity) one arrives at the orthogonality relation

$$\int_{\mathbb{R}} \frac{F_m(w;q,x)F_n(w;q,x)}{\gamma(\psi_w(x) - w^2\phi_w(x))^2 + \gamma^{-1}(1-w^2)^2\phi_w^2(x)}\,dx = \frac{\pi}{(1-w^2)^2} w^{-2n} q^{-n^2} \delta_{mn},$$

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  for $1 \neq w^{-2} \in (q, q^{-1})$.
- The orthogonality relation for $w = 1$ reads

$$\int_{\mathbb{R}} \frac{F_m(1;q,x)F_n(1;q,x)}{\gamma(x\phi_1'(x) - \phi_1(x))^2 + \gamma^{-1}(\phi_1(x))^2}\,dx = \pi q^{-n^2} \delta_{mn}.$$

Contents

The moment sequence

- We denote $a := w^{-2}$ and

$$m_n = m_n(a;q) := \int_{\mathbb{R}} x^n d\mu^{(a;q)}(x), \quad n \in \mathbb{Z}_+,$$

where $\mu^{(a;q)}$ is a measure of orthogonality for q-Lommel polynomials normalized to $m_0 = 1$.

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- It seems the moment sequence m_n can not be expressed explicitly.

The moment sequence

- We denote $a := w^{-2}$ and
$$m_n = m_n(a;q) := \int_{\mathbb{R}} x^n d\mu^{(a;q)}(x), \quad n \in \mathbb{Z}_+,$$
where $\mu^{(a;q)}$ is a measure of orthogonality for q-Lommel polynomials normalized to $m_0 = 1$.
- It seems the moment sequence m_n can not be expressed explicitly.
- It would be of interest to know the asymptotic behavior of m_n, for $n \to \infty$, in particular in the case of indeterminate Hamburger moment problem, i.e., $q < a < 1/q$.

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Quadratic recursion

$$m_{n+2}(a;q) = (a+1)m_{n+1}(a;q) + \frac{a}{q}\sum_{k=0}^{n} q^{-k} m_k(a;q)m_{n-k}(a;q), \qquad n \in \{-1,0,1,2,\dots\}.$$

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Linear recursion

$$m_n(a;q) = \frac{\omega_n(a;q)}{(q;q)_{n-1}} - \sum_{k=1}^{n-1} \frac{q^k}{(q;q)_k}\omega_k(a;q) m_{n-k}(a;q), \qquad n \in \mathbb{N}.$$

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- Consequently,

$$m_n(a;q) \leq \frac{\omega_n(a;q)}{(q;q)_{n-1}} \leq \frac{(1+a)^n}{(q;q)_{n-1}} q^{-\frac{n^2}{4}}, \quad n \in \mathbb{N}.$$

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Gracia!