The Characteristic Function for Jacobi Matrices with Applications

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Combinatorics on Words and Mathematical Physics

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Outline

- Consider Jacobi operator J acting on vectors from standard basis $\{e_n\}_{n=1}^{\infty}$ of $\ell^2(\mathbb{N})$ as

$$Je_n = w_{n-1}e_{n-1} + \lambda_n e_n + w_n e_{n+1} \quad (w_0 := 0)$$

where $\lambda_n \in \mathbb{C}$, $w_n \in \mathbb{C} \setminus \{0\}$, and $n \in \mathbb{N}$.

Motivation - introduction

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$$\mathrm{Dom}(J) = \{x \in \ell^2(\mathbb{N}) : Jx \in \ell^2(\mathbb{N})\}.$$

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- The matrix representation of J in the standard basis:
$$J = \begin{pmatrix} \lambda_1 & w_1 & & & \\ w_1 & \lambda_2 & w_2 & & \\ & w_2 & \lambda_3 & w_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

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- Objective: Investigation of the spectrum of J when the diagonal sequence dominates the off-diagonal in some sense.

Motivation - reformulation of the problem

For $z \in \mathbb{C}$ and $\lambda_n > 0$ define

$$A(z) := L^{-1/2}(UW + WU^* - z)L^{-1/2} = \begin{pmatrix} -\frac{z}{\lambda_1} & \frac{w_1}{\sqrt{\lambda_1\lambda_2}} & & & \\ \frac{w_1}{\sqrt{\lambda_1\lambda_2}} & -\frac{z}{\lambda_2} & \frac{w_2}{\sqrt{\lambda_2\lambda_3}} & & \\ & \frac{w_2}{\sqrt{\lambda_2\lambda_3}} & -\frac{z}{\lambda_3} & \frac{w_3}{\sqrt{\lambda_3\lambda_4}} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

where $L = \operatorname{diag}(\lambda_1, \lambda_2, \dots)$, $W = \operatorname{diag}(w_1, w_2, \dots)$, and U is unilateral shift.

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Assertion

Let $A(z)$ be Hilbert-Schmidt operator for some $0 \neq z \in \mathbb{C}$. Then

$$z \in \rho(J) \quad \text{iff} \quad -1 \in \rho(A(z))$$

and it holds

$$(J - z)^{-1} = L^{-1/2}(1 + A(z))^{-1}L^{-1/2}.$$

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To investigate the spectrum of J one can consider operator $A(z)$ instead. Main advantages are:

- $A(z)$ is Hilbert-Schmidt, while J is unbounded
- one can use function $z \mapsto \det_2(1 + A(z))$ which is well defined as an entire function.

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Let me define $\mathfrak{F} : D \to \mathbb{C}$ by relation

$$\mathfrak{F}(x) = 1 + \sum_{m=1}^{\infty}(-1)^m \sum_{k_1=1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \dots \sum_{k_m=k_{m-1}+2}^{\infty} x_{k_1} x_{k_1+1} x_{k_2} x_{k_2+1} \dots x_{k_m} x_{k_m+1},$$

where

$$D = \left\{ \{x_k\}_{k=1}^{\infty} \subset \mathbb{C};\ \sum_{k=1}^{\infty} |x_k x_{k+1}| < \infty \right\}.$$

For a finite number of complex variables let me identify $\mathfrak{F}(x_1, x_2, \dots, x_n)$ with $\mathfrak{F}(x)$ where $x = (x_1, x_2, \dots, x_n, 0, 0, 0, \dots)$.

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- Note that the domain D is not a linear space. One has, however, $\ell^2(\mathbb{N}) \subset D$.

Properties of $\mathfrak{F}$

- For all $x \in D$ and $k = 1, 2, \ldots$ one has

Recursive relation

$$\mathfrak{F}(x) = \mathfrak{F}(x_1, \ldots, x_k)\,\mathfrak{F}(T^k x) - \mathfrak{F}(x_1, \ldots, x_{k-1}) x_k x_{k+1} \mathfrak{F}(T^{k+1} x)$$

where T denotes the truncation operator from the left defined on the space of all sequences:

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- Functions $\mathfrak{F}$ restricted on $\ell^2(\mathbb{N})$ is a continuous functional on $\ell^2(\mathbb{N})$. Further, for $x \in D$, it holds

$$\lim_{n\to\infty} \mathfrak{F}(x_1, x_2, \dots, x_n) = \mathfrak{F}(x) \qquad \text{and} \qquad \lim_{n\to\infty} \mathfrak{F}(T^n x) = 1.$$

Other properties of $\mathfrak{F}$

- Initial values $\mathfrak{F}(\emptyset) = \mathfrak{F}(x_1) = 1$ together with relation

$$\mathfrak{F}(x_1, \ldots, x_{n-1}, x_n) = \mathfrak{F}(x_1, \ldots, x_{n-2}, x_{n-1}) - x_{n-1} x_n \, \mathfrak{F}(x_1, \ldots, x_{n-3}, x_{n-2})$$

  determine recursively and unambiguously $\mathfrak{F}(x_1, \ldots, x_n)$ for any finite number of variables.

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- Other equivalent definitions of $\mathfrak{F}(x_1, x_2, \dots, x_n)$ is:

$$\mathfrak{F}(x_1, x_2, \dots, x_n) = \det X_n = \det \begin{pmatrix} 1 & x_1 & & & \\ x_2 & 1 & x_2 & & \\ & \ddots & \ddots & \ddots & \\ & & x_{n-1} & 1 & x_{n-1} \\ & & & x_n & 1 \end{pmatrix}.$$

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- Function $\mathfrak{F}$ is related to a continued fraction. For a given $x \in D$ such that $\mathfrak{F}(x) \neq 0$, it holds

$$\frac{\mathfrak{F}(Tx)}{\mathfrak{F}(x)} = \cfrac{1}{1 - \cfrac{x_1 x_2}{1 - \cfrac{x_2 x_3}{1 - \cfrac{x_3 x_4}{1 - \dots}}}}.$$

Proposition

Let $\{\lambda_n\}$ be positive and

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \left| \frac{w_n^2}{\lambda_n \lambda_{n+1}} \right| < \infty.$$

Then $A(z)$ is Hilbert-Schmidt for all $z \in \mathbb{C}$ and it holds

$$\det{}_2(1 + A(z)) = \mathfrak{F}\left(\left\{ \frac{\gamma_n^2}{\lambda_n - z} \right\}_{n=1}^{\infty} \right) \prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n} \right) e^{z/\lambda_n}$$

where the sequence $\{\gamma_n\}$ can be defined recursively as $\gamma_1 = 1$, $\gamma_{k+1} = w_k / \gamma_k$.

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- In the following we focus just on the function

$$F_J(z) := \mathfrak{F}\left(\left\{ \frac{\gamma_n^2}{\lambda_n - z} \right\}_{n=1}^{\infty} \right).$$

- Function F_J is well defined on $\mathbb{C} \setminus \overline{\{\lambda_n\}}$ if

$$\left\{ \frac{\gamma_n^2}{\lambda_n - z} \right\}_{n=1}^{\infty} \in D \quad \text{for all } z \in \mathbb{C} \setminus \overline{\{\lambda_n\}}$$

which holds if there is at least one $z_0 \in \mathbb{C} \setminus \overline{\{\lambda_n\}}$ such that

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(λ_n and w_n are complex!)

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- F_J is meromorphic function on $\mathbb{C} \setminus \overline{\{\lambda_n\}}$ with poles in $z \in \{\lambda_n\} \setminus \mathrm{der}(\{\lambda_n\})$ of finite order less or equal to the number

$$r(z) := \sum_{n=1}^{\infty} \delta_{z,\lambda_n}.$$

Definition

Let us define

$$\mathfrak{Z}(J) := \left\{ z \in \mathbb{C} \setminus \operatorname{der}(\lambda);\ \lim_{u \to z} (u-z)^{r(z)} F_J(u) = 0 \right\}$$

and, for $k \in \mathbb{Z}_+$ and $z \in \mathbb{C} \setminus \operatorname{der}(\{\lambda_n\})$, we put

$$\xi_k(z) := \lim_{u \to z} (u-z)^{r(z)} \left(\prod_{l=1}^{k} \frac{w_{l-1}}{u - \lambda_l} \right) \mathfrak{F}\left(\left\{ \frac{\gamma_l^2}{\lambda_l - u} \right\}_{l=k+1}^{\infty} \right)$$

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- Note that for $z \in \mathbb{C} \setminus \overline{\{\lambda_n\}}$, $\xi_0(z) = F_J(z)$.
- We call $\xi_0(z) \equiv \lim_{u \to z} (u - z)^{r(z)} F_J(u)$ the *characteristic function* of Jacobi matrix J.

Proposition

If $\xi_0(z) = 0$ for some $z \in \mathbb{C} \setminus \mathrm{der}(\{\lambda_n\})$, then z is an eigenvalue of J and

$$\xi(z) := (\xi_1(z), \xi_2(z), \xi_3(z), \dots)$$

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- Moreover, for $z \notin \overline{\{\lambda_n\}}$, vector $\xi(z)$ satisfies the formula

$$\sum_{k=1}^{\infty} (\xi_k(z))^2 = \xi_0'(z)\xi_1(z) - \xi_0(z)\xi_1'(z).$$

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- Consequently, if $\{\lambda_n\}$ and $\{w_n\}$ are real sequences and $z \in \mathfrak{Z}(J) \setminus \{\lambda_n\}$ then

$$\|\xi(z)\|^2 = \xi_0'(z)\xi_1(z).$$

Proposition

If $z \notin (\mathfrak{Z}(J) \cup \operatorname{der}(\{\lambda_n\}))$ then $z \in \rho(J)$. Consequently, it holds

$$\operatorname{spec}(J) \setminus \operatorname{der}(\{\lambda_n\}) = \operatorname{spec}_p(J) \setminus \operatorname{der}(\{\lambda_n\}) = \mathfrak{Z}(J).$$

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$$G_{ij}(z) = (e_i, (J-z)^{-1} e_j) = -\frac{1}{w_M} \prod_{l=m}^{M} \left(\frac{w_l}{z-\lambda_l}\right) \frac{\mathfrak{F}\left(\left\{\frac{\gamma_l^2}{\lambda_l - z}\right\}_{l=1}^{m-1}\right) \mathfrak{F}\left(\left\{\frac{\gamma_l^2}{\lambda_l - z}\right\}_{l=M+1}^{\infty}\right)}{\mathfrak{F}\left(\left\{\frac{\gamma_l^2}{\lambda_l - z}\right\}_{l=1}^{\infty}\right)}$$

where $m := \min(i, j)$ and $M := \max(i, j)$.

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Especially, we get a compact formula for the Weyl m-function $m(z) = G_{11}(z)$,

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Proposition

Let $x = \{x_n\}_{n=1}^{\infty} \subset \mathbb{C} \setminus \{0\}$ satisfies $\sum_n |x_n x_{n+1}| < \infty$ and $\mathfrak{F}(x) \neq 0$ then any solution of recurrence

$$F_n - F_{n+1} + x_n x_{n+1} F_{n+2} = 0, \quad n \in \mathbb{N}. \tag{1}$$

is a linear combination of solutions

$$F_n := \mathfrak{F}(T^{n-1}x) = \mathfrak{F}\left(\{x_k\}_{k=n}^{\infty}\right), \quad n \in \mathbb{N}$$

and

$$G_n := \left(\prod_{k=1}^{n-2} \frac{1}{x_k x_{k+1}}\right) \mathfrak{F}\left(\{x_k\}_{k=1}^{n-2}\right), \quad n \in \{2, 3, \dots\}, \quad G_1 := 0.$$

Moreover, solution F is the unique solution of (1) satisfying boundary condition $\lim_{n\to\infty} F_n = 1$.

Bessel functions

Let $w, \alpha \in \mathbb{C}$, $z - r\alpha \notin \alpha\mathbb{N}$, and $r \in \mathbb{Z}_+$ then it holds

$$\mathfrak{F}\left(\left\{\frac{w}{\alpha k - z}\right\}_{k=r+1}^{\infty}\right) = \left(\frac{w}{\alpha}\right)^{-r+z/\alpha} \Gamma\left(1 + r - \frac{z}{\alpha}\right) J_{r-z/\alpha}\left(\frac{2w}{\alpha}\right).$$

For $r = 0$, the above function is characteristic function form Jacobi operator J of the form

$$J = \begin{pmatrix} \alpha & w & & & \\ w & 2\alpha & w & & \\ & w & 3\alpha & w & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

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$$\mathrm{spec}(J) = \{z \in \mathbb{C};\ J_{-\frac{z}{\alpha}}\left(\frac{2w}{\alpha}\right) = 0\},$$

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and the formula for the kth entry of the respective eigenvector is

$$v_k(z) = (-1)^k J_{k-\frac{z}{\alpha}}\left(\frac{2w}{\alpha}\right).$$

- For $w, \nu \in \mathbb{C}$, $\nu + n \notin -\mathbb{Z}_+$, $0 < q < 1$, and $n \in \mathbb{Z}$, it holds

$$\mathfrak{F}\left(\left\{\frac{w}{[\nu+k]_q}\right\}_{k=n}^{\infty}\right) = {}_0\phi_1(;q^{\nu+n};q,-w^2(1-q)^2q^{\nu+n-\frac{1}{2}})$$

where $[\alpha]_q$ stands for q-deformed number, i.e.,

$$[\alpha]_q := \frac{q^{\frac{\alpha}{2}} - q^{-\frac{\alpha}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}.$$

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- By using definitions

$$J_\nu(x;q) := \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\left(\frac{x}{2}\right)^\nu {}_0\phi_1\left(;q^{\nu+1};q,-\frac{x^2}{4}q^{\nu+1}\right)$$

and

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x},$$

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q-Bessel functions

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$$\mathfrak{F}\left(\left\{\frac{w}{[\nu+k]_q}\right\}_{k=1}^{\infty}\right) = \Gamma_q(\nu+1)(wq^{-\frac{1}{4}})^{-\nu} J_\nu(2w(1-q)q^{-\frac{1}{4}}; q).$$

Confluent Hypergeometric Function ${}_1F_1$

For $\mu, \nu, z \in \mathbb{C}$, $\mu - 1 \notin \frac{1}{2}\mathbb{Z}_+$, confluent hypergeometric function ${}_1F_1$ satisfies the three term recurrence of the form

$$\begin{aligned} {}_1F_1(\mu+\nu-1; 2\mu-2; 2z) &= \left(1 + \frac{\nu z}{\mu(\mu-1)}\right) {}_1F_1(\mu+\nu; 2\mu; 2z) \\ &+ \frac{z^2(\mu^2-\nu^2)}{\mu^2(4\mu^2-1)} {}_1F_1(\mu+\nu+1; 2\mu+2; 2z). \end{aligned}$$

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From this, one can verify, the function

$$F_n := e^{-z} \prod_{k=n}^{\infty} \left(1 + \frac{\nu z}{(\mu+k)(\mu+k+1)}\right)^{-1} {}_1F_1(\mu+n+\nu; 2\mu+2n; 2z)$$

fulfills $\lim_{n\to\infty} F_n = 1$ together with the recurrence rule

$$F_n - F_{n+1} + \frac{w_n^2}{(1/z+\lambda_n)(1/z+\lambda_{n+1})} F_{n+2} = 0$$

where

$$\lambda_n = \frac{\nu}{(\mu+n)(\mu+n+1)}$$

and

$$w_n^2 = \frac{\nu^2 - (\mu+n+1)^2}{(\mu+n+1)^2(4(\mu+n+1)^2-1)}.$$

By the proposition on the uniqueness of the solution the recurrence equations one gets identity

$$\mathfrak{F}\left(\left\{\frac{\gamma_k^2}{\lambda_k + 1/z}\right\}_{k=n}^{\infty}\right) = e^{-z}\prod_{k=n}^{\infty}\left(1+\frac{\nu z}{(\mu+k)(\mu+k+1)}\right)^{-1} {}_1F_1(\mu+n+\nu; 2\mu+2n; 2z)$$

where, for $n \in \mathbb{Z}$, one has to set

$$\lambda_n := \frac{\nu}{(\mu+n)(\mu+n+1)}$$

and

$$w_n := \frac{i}{\mu+n+1}\sqrt{\frac{(\mu+n+1)^2-\nu^2}{(2\mu+2n+1)(2\mu+2n+3)}}.$$

Parameters $\mu, \nu \in \mathbb{C}$ are restricted as follows: $2\mu + 2n \notin -\mathbb{Z}_+$ and $|\mu + k| \neq |\nu|$ for $k - n \in \mathbb{N}$.

- The regular Coulomb wave function $F_L(\eta, \rho)$ is one of two linearly independent solutions of the second-order differential equation

$$\frac{d^2u}{d\rho^2} + \left[1 - \frac{2\eta}{\rho} - \frac{L(L+1)}{\rho^2}\right] u = 0$$

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- $F_L(\eta, \rho)$ can be decomposed as follows,

$$F_L(\eta, \rho) = C_L(\eta)\rho^{L+1}\phi_L(\eta, \rho)$$

where

$$C_L(\eta) = \sqrt{\frac{2\pi\eta}{e^{2\pi\eta} - 1}} \frac{\sqrt{(1+\eta^2)(4+\eta^2)\dots(L^2+\eta^2)}}{(2L+1)!!L!}$$

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- Hence one can use the relation between $\mathfrak{F}$ and ${}_1F_1$ to find the following formula.

Proposition

For $\eta \in \mathbb{C}$, $\rho \in \mathbb{C} \setminus \{0\}$, $\eta\rho \neq -k(k+1)$, $k \geq n+1$, and $n \in \mathbb{Z}_+$, one has

$$\mathfrak{F}\left(\left\{\frac{\gamma_k^2}{\lambda_k + 1/\rho}\right\}_{k=n+1}^{\infty}\right) = \frac{\pi\eta\rho}{\cos\left(\frac{\pi}{2}\sqrt{1-4\eta\rho}\right)} \prod_{k=1}^{n}\left[1 + \frac{\eta\rho}{k(k+1)}\right]\phi_n(\eta,\rho).$$

The entry sequences now reads

$$w_n = \frac{\sqrt{(n+1)^2+\eta^2}}{(n+1)\sqrt{(2n+1)(2n+3)}} \quad \text{and} \quad \lambda_n := \frac{\eta}{n(n+1)}.$$

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Consequently, for corresponding Jacobi matrix

$$J_L = \begin{pmatrix} -\lambda_{L+1} & w_{L+1} & & & \\ w_{L+1} & -\lambda_{L+2} & w_{L+2} & & \\ & w_{L+2} & -\lambda_{L+3} & w_{L+3} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

we get

$$\operatorname{spec}(J_L) = \{1/\rho : \phi_L(\eta,\rho) = 0\} \cup \{0\} = \{1/\rho : F_L(\eta,\rho) = 0\} \cup \{0\}$$

and

$$v(1/\rho) = \left(\sqrt{2L+3}F_{L+1}(\eta,\rho), \sqrt{2L+5}F_{L+2}(\eta,\rho), \sqrt{2L+7}F_{L+3}(\eta,\rho), \dots\right)^T.$$

Proposition

For $\delta, a \in \mathbb{C}$, and $n \in \mathbb{Z}_+$, it holds

$$\mathfrak{F}\left(\left\{\frac{\gamma_k^2}{(a+1)q^{k-1}-z}\right\}_{k=n+1}^{\infty}\right) = \frac{(z^{-1}q^n;q)_\infty}{((a+1)z^{-1}q^n;q)_\infty}\,{}_1\phi_1\left(z^{-1}q^\delta, z^{-1}q^n; q, az^{-1}q^n\right)$$

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- For $a>0$, the operator J is not hermitian, however, $\operatorname{spec}(J)$ is real!

- For $\lambda_n \in \mathbb{R}$ and $w_n > 0$, OPs can be defined recursively by

$$w_{n-1}y_{n-1}(x) + \lambda_n y_n(x) + w_n y_{n+1}(x) = xy_n(x), \quad n = 1, 2, \ldots \quad (w_0 := -1)$$

and OPs of the first kind $P_n(x)$ satisfy initial conditions $P_0(x) = 0$, $P_1(x) = 1$, while OPs of the second kind $Q_n(x)$ satisfy $Q_0(x) = 1$, $Q_1(x) = 0$.

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- OPs are related to $\mathfrak{F}$ through identities

$$P_{n+1}(z) = \prod_{k=1}^{n} \left(\frac{z-\lambda_k}{w_k}\right) \mathfrak{F}\left(\left\{\frac{\gamma_l^2}{\lambda_l - z}\right\}_{l=1}^{n}\right), \quad n = 0, 1 \ldots,$$

$$Q_{n+1}(z) = \frac{1}{w_1} \prod_{k=2}^{n} \left(\frac{z-\lambda_k}{w_k}\right) \mathfrak{F}\left(\left\{\frac{\gamma_l^2}{\lambda_l - z}\right\}_{l=2}^{n}\right), \quad n = 0, 1 \ldots$$

Proposition

Let J be self-adjoint and either J has discrete spectrum or it is a compact operator. Then, for $m, n \in \mathbb{N}$, the orthogonality relation

$$\sum_{\lambda \in \mathfrak{Z}(J)} \frac{F_{J,2}(\lambda)}{(\lambda - \lambda_1) F_J'(\lambda)} P_n(\lambda) P_m(\lambda) = \delta_{m,n}$$

holds, where $F_{J,k+1}$ is the characteristic function of the Jacobi operator defined by using shifted sequences $\{\lambda_{n+k}\}_{n=1}^{\infty}$ and $\{w_{n+k}\}_{n=1}^{\infty}$, i.e.,

$$F_{J,k+1}(z) = \mathfrak{F}\left(\left\{ \frac{\gamma_l^2}{\lambda_l - z} \right\}_{l=k}^{\infty} \right), \quad (F_{J,1} = F_J).$$

Show the Askey Scheme

- Explicit formula:

$$R_{n,\nu}(x) = \sum_{k=0}^{[n/2]} \binom{n-k}{k} (-1)^k \frac{\Gamma(\nu+n-k)}{\Gamma(\nu+k)} \left(\frac{2}{x}\right)^{n-2k}$$

Well known results on Lommel polynomials

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- Relation to $\mathfrak{F}$:

$$R_{n,\nu}(x) = \left(\frac{2}{x}\right)^n \frac{\Gamma(\nu+n)}{\Gamma(\nu)} \mathfrak{F}\left(\left\{\frac{x}{2(\nu+k)}\right\}_{k=0}^{n-1}\right)$$

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- Due to general identity

$$\mathfrak{F}(x_1,\ldots,x_n)\mathfrak{F}(Tx) - \mathfrak{F}(x_2,\ldots,x_n)\mathfrak{F}(x) = \left(\prod_{k=1}^{n} x_k x_{k+1}\right) \mathfrak{F}(T^{n+1}x),$$

which holds for any $x \in D$, one can rederive the well-known relation between Lommel polynomials and Bessel functions,

$$R_{n,\nu}(x)J_\nu(x) - R_{n-1,\nu+1}(x)J_{\nu-1}(x) = J_{\nu+n}(x)$$

where $n \in \mathbb{N}$, and $\nu, x \in \mathbb{C}$.

- Explicit formula:

$$R_{n,\nu}(x) = \sum_{k=0}^{[n/2]} \binom{n-k}{k} (-1)^k \frac{\Gamma(\nu+n-k)}{\Gamma(\nu+k)} \left(\frac{2}{x}\right)^{n-2k}$$

- Relation to $\mathfrak{F}$:

$$R_{n,\nu}(x) = \left(\frac{2}{x}\right)^n \frac{\Gamma(\nu+n)}{\Gamma(\nu)} \mathfrak{F}\left(\left\{\frac{x}{2(\nu+k)}\right\}_{k=0}^{n-1}\right)$$

- Due to general identity

$$\mathfrak{F}(x_1,\ldots,x_n)\mathfrak{F}(Tx) - \mathfrak{F}(x_2,\ldots,x_n)\mathfrak{F}(x) = \left(\prod_{k=1}^{n} x_k x_{k+1}\right)\mathfrak{F}(T^{n+1}x),$$

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- OG relation:

$$\sum_{k\in\pm\mathbb{N}} x_{k,\nu}^{-2} R_{n,\nu+1}(x_{k,\nu}) R_{m,\nu+1}(x_{k,\nu}) = \frac{2}{n+1+\nu}\delta_{mn},$$

  for $\nu > -1$ and $m, n \in \mathbb{Z}_+$.

- Let

$$w_n := \frac{\sqrt{(n+1)^2+\eta^2}}{(n+1)\sqrt{(2n+1)(2n+3)}} \quad \text{and} \quad \lambda_n := \frac{\eta}{n(n+1)}.$$

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- For $\eta \in \mathbb{R}$, $L \in \mathbb{Z}_+$, define the set of OG polynomials $\{P_n^{(L)}(\eta; z)\}_{n=0}^{\infty}$ by recurrence rule

$$zP_n^{(L)}(\eta; z) = w_{n-1+L}P_{n-1}^{(L)}(\eta; z) - \lambda_{n+L}P_n^{(L)}(\eta; z) + w_{n+L}P_{n+1}^{(L)}(\eta; z)$$

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$$O_{n+1}^{(L-1)}(\eta;\rho)F_L(\eta,\rho) - O_n^{(L)}(\eta;\rho)F_{L-1}(\eta,\rho) = \frac{L}{\sqrt{L^2+\eta^2}}F_{L+n}(\eta,\rho)$$

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$$\sum_{\rho_{\eta,L}} \rho_{\eta,L}^{-2} R_n^{(L)}(\eta;\rho_{\eta,L}) R_m^{(L)}(\eta;\rho_{\eta,L}) = \frac{(L+1)^2+\eta^2}{(2L+3)(L+1)^2}\delta_{mn}$$

where $m, n \in \mathbb{N}$, $\eta \in \mathbb{R}$, and $L \in \mathbb{Z}_+$. The summation is over the set of all nonzero roots $\rho_{\eta,L}$ of $F_L(\eta,\rho)$.

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- Rodrigez type formula for $R_n^{(L)}(\eta;\rho)$: ?

Thank you!