The Moment Problem

František Štampach

Seminar talk - analysis group

November 2, 2016
Outline

1. Motivation
2. What the moment problem is?
3. Existence and uniqueness of the solution - operator approach
4. Jacobi matrix and Orthogonal Polynomials
5. Sufficient conditions for determinacy
6. The set of solutions of indeterminate moment problem
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Motivation

Chebychev’s question (1874, most likely inspired by work of I. J. Bienaymé, 1853):
*If for some positive function* $f$,

$$\int_{\mathbb{R}} x^n f(x) \, dx = \int_{\mathbb{R}} x^n e^{-x^2} \, dx, \quad \forall n = 0, 1, \ldots$$

*can we then conclude that* $f(x) = e^{-x^2}$? 

In today’s language:

*Is the normal density uniquely determined by its moment sequence?*

Answer: *yes* in the sense that $f(x) = e^{-x^2}$ a.e. w.r.t. Lebesgue measure on $\mathbb{R}$.

Relevant questions immediately appear:

*What happens if one replaces the normal density by something else?*

The general answer to Chebychev’s question is *no*. Suppose, e.g., $X \sim N(0, \sigma^2)$ and consider densities of $\exp(X)$ (lognormal distribution), then we lost the uniqueness.

And what if one replaces the RHS by a sequence of real numbers $s_n$? Does there even exist a distribution (measure) whose $n$-th moment is equal to $s_n$?

Answer: In general, *no*. 

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Relevant questions immediately appear: *What happens if one replaces the normal density by something else?* 

The general answer to Chebychev’s question is *no*. Suppose, e.g., $X \sim \mathcal{N}(0, \sigma^2)$ and consider densities of $\exp(X)$ (lognormal distribution), then we lost the uniqueness. 

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What is the moment problem

Let $I \subset \mathbb{R}$ be a closed interval. For a positive measure $\mu$ on $I$ the $n$th moment is defined as

$$\int_I x^n \, d\mu(x),$$

(provided the integral exists).

Suppose a real sequence $\{s_n\}_{n \geq 0}$ is given. The moment problem on $I$ consists of solving the following three problems:

1. Does there exist a positive measure on $I$ with moments $\{s_n\}_{n \geq 0}$?
2. If so, is this positive measure uniquely determined by moments $\{s_n\}_{n \geq 0}$? (determinate case)
3. If this is not the case, how one can describe all positive measures on $I$ with moments $\{s_n\}_{n \geq 0}$? (indeterminate case)

One can restrict oneself to cases:

- $I = \mathbb{R} -$ Hamburger moment problem ($M_H = \text{set of solutions}$)
- $I = [0, +\infty)$ - Stieltjes moment problem ($M_S = \text{set of solutions}$)
- $I = [0, 1]$ - Hausdorff moment problem
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- uniqueness $\sim$ determinate case vs. non-uniqueness $\sim$ indeterminate case
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uniqueness $\preceq$ *determinate case* vs. non-uniqueness $\succeq$ *indeterminate case*.

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- $I = [0, 1]$ - *Hausdorff* moment problem
Hausdorff moment problem

Theorem (Hausdorff, 1923)

The moment problem has a solution on $[0, 1]$ iff sequence $\{s_n\}_{n \geq 0}$ is completely monotonic, i.e.,

$$(-1)^k (\Delta^k s)_n \geq 0$$

for all $k, n \in \mathbb{Z}_+$, where $(\Delta s)_n = s_{n+1} - s_n$. 

and moreover ...

The Hausdorff moment problem is always determinate. Further, we will discuss the Stieltjes and Hamburger moment problem only...
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(H_N(s))_{ij} := s_{i+j}, \quad i, j \in \{0, 1, \ldots N - 1\}.
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Existence of the solution (necessary condition)

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- Define two sesquilinear forms \( H_N \) and \( S_N \) on \( \mathbb{C}^N \) by
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  H_N(x, y) := \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \overline{x_i} y_j s_{i+j} \quad \text{and} \quad S_N(x, y) := \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \overline{x_i} y_j s_{i+j+1}.
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Hence \( H_N(x, y) = (x, H_N(s)y) \) and \( S_N(x, y) = (x, H_N(Ts)y) \) (\( . , . \) Euclidean inner product).
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- Let \( \mu \in \mathcal{M}_H \) or \( \mu \in \mathcal{M}_S \) with infinite support. By observing that
  \[
  H_N(y, y) = \int \left| \sum_{i=0}^{N-1} y_i x^i \right|^2 d\mu(x) \quad \text{and} \quad S_N(y, y) = \int x \left| \sum_{i=0}^{N-1} y_i x^i \right|^2 d\mu(x),
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  one immediately gets the following.
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**Necessary condition for the existence:**

A necessary condition for the Hamburger moment problem to have a solution (with infinite support) is the sesquilinear form \( H_N \) is PD for all \( N \in \mathbb{Z}_+ \). A necessary condition for the Stieltjes moment problem to have a solution (with infinite support) is both sesquilinear forms \( H_N \) and \( S_N \) are PD for all \( N \in \mathbb{Z}_+ \).
Existence and uniqueness of the solution - operator approach

Existence of the solution (sufficient condition)

Let \( \{s_n\}_{n \geq 0} \) is given such that \( H_N(s) \) are PD for all \( N \in \mathbb{N} \).
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- Let $\{s_n\}_{n \geq 0}$ is given such that $H_N(s)$ are PD for all $N \in \mathbb{N}$.
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- Let \( \mathbb{C}[x] \) be the ring of complex polynomials.
- For \( P, Q \in \mathbb{C}[x] \),

\[
P(x) = \sum_{k=0}^{N-1} a_k x^k, \quad \text{and} \quad Q(x) = \sum_{k=0}^{N-1} b_k x^k,
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define positive definite inner product

\[
\langle P, Q \rangle := H_N(a, b).
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- In particular,

\[
\langle 1, A^n 1 \rangle = \langle 1, x^n \rangle = s_n, \quad n \in \mathbb{N}_0.
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Existence of the solution (sufficient condition)

- $A$ has a self-adjoint extension since it commutes with a complex conjugation operator $C$ on $\mathbb{C}[x]$ (von Neumann).
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- Hence, for a suitable function $f$, it holds

$$\langle 1, f(A')1 \rangle = \int_{\mathbb{R}} f(x) d\mu(x).$$
Existence of the solution (sufficient condition)

- \( A \) has a self-adjoint extension since it commutes with a complex conjugation operator \( C \) on \( \mathbb{C}[x] \) (von Neumann).

- If each \( S_N \) is PD, then

\[
\langle P, A[P] \rangle = S_N (a, a) \geq 0, \quad \forall P \in \mathbb{C}[x],
\]

and it follows \( A \) has a non-negative self-adjoint extension \( A_F \), the Friedrichs extension.

- Let \( A' \) be a self-adjoint extension of \( A \). By the spectral theorem there is a projection valued spectral measure \( E_{A'} \) and positive measure

\[
\mu(\cdot) = \langle 1, E_{A'}(\cdot) 1 \rangle.
\]

- Hence, for a suitable function \( f \), it holds

\[
\langle 1, f(A') 1 \rangle = \int_{\mathbb{R}} f(x) d\mu(x).
\]

- Especially, for \( f(x) = x^n \), one finds

\[
s_n = \langle 1, A^n 1 \rangle = \langle 1, (A')^n 1 \rangle = \int_{\mathbb{R}} x^n d\mu(x).
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Existence of the solution

- We see a self-adjoint extension of $A$ yields a solution of the Hamburger moment problem.
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- If, additionally, each $S_N$ is PD, $A_F$ is a non-negative self-adjoint extension of $A$ and for the corresponding measure one has $\text{supp}(\mu) \subset [0, \infty)$. So there is a solution of the Stieltjes moment problem.
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Theorem (existence):

i) A necessary and sufficient condition for $\mathcal{M}_H \neq \emptyset$ (with infinite support) is

$$\det H_N(s) > 0 \quad \forall N \in \mathbb{N}.$$ 

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- Historically, this result has not been obtained by using the spectral theorem that was invented later.
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Uniqueness

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Theorem (uniqueness):

i) A necessary and sufficient condition for the Hamburger moment problem to be determinate is that the operator $A$ is essentially self-adjoint (i.e., it has a unique self-adjoint extension).

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Existence and uniqueness of the solution - operator approach

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The other direction is even less clear. For not only is it not obvious, it is false that every solution of the moment problem arise from some measure given by spectral measure of some self-adjoint extension.
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**Theorem (uniqueness):**

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In one direction, it is not clear that distinct self-adjoint extensions $A_1'$ and $A_2'$ give rise to distinct measures $\mu_1$ and $\mu_2$.

The other direction is even less clear. For not only is it not obvious, it is **false** that every solution of the moment problem arise from some measure given by spectral measure of some self-adjoint extension.

A solution of the moment problem which comes from a self-adjoint extension of $A$ is called $N$-extremal solution (von Neumann [Simon], extremal [Shohat–Tamarkin]).
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By applying the Gramm-Schmidt procedure, we obtain an orthonormal basis $\{P_n\}_{n=0}^{\infty}$ of $\mathcal{H}^{(s)}$.

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By construction, $P_n$ is a polynomial of degree $n$ with real coefficients and

\[
\langle P_m, P_n \rangle = \delta_{mn}, \quad \forall m, n \in \mathbb{N}_0.
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\( \{P_n\}_{n=0}^\infty \) are determined by moment sequence \( \{s_n\}_{s=0}^\infty \),

\[
P_n(x) = \frac{1}{\sqrt{\det[H_{n+1}(s)H_n(s)]}} \begin{vmatrix} s_0 & s_1 & \ldots & s_n \\ s_1 & s_2 & \ldots & s_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n-1} & s_n & \ldots & s_{2n-1} \\ 1 & x & \ldots & x^n \end{vmatrix}.
\]
Since

\[ \text{span}(1, x, \ldots, x^n) = \text{span}(P_0, P_1, \ldots, P_n), \]

the polynomial \( xP_n(x) \) has an expansion in \( P_0, P_1, \ldots, P_{n+1} \).
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Moreover, if \( 0 \leq j < n - 1 \), one has
\[ \langle P_j, xP_n \rangle = \langle xP_j, P_n \rangle = 0. \]

There are sequences \( \{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}, \text{ and } \{c_n\}_{n=0}^{\infty} \) such that
\[ xP_n(x) = c_n P_{n+1}(x) + b_n P_n(x) + a_{n-1} P_{n-1}(x), \quad (P_{-1}(x) := 0), \]
for \( n \in \mathbb{N}_0 \).
Since

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Thus, any sequence of orthogonal polynomials satisfies a three-term recurrence
\[ xP_n(x) = a_n P_{n+1}(x) + b_n P_n(x) + a_{n-1} P_{n-1}(x) \]
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And \( A \) has, in the basis \( \{P_n\}_{n=0}^{\infty} \), a symmetric tridiagonal matrix representation.
Under the unitary mapping

\[ U : \mathcal{H}^{(s)} \to \ell^2(\mathbb{N}_0) : P_n \mapsto e_n \]

the operator \( A \) is transformed to the operator \( U^* AU \) which we denote again by \( A \) only.
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\[ A = \begin{pmatrix} b_0 & a_0 & & \\ a_1 & b_1 & a_1 & \\ a_2 & b_2 & b_3 & \\ & & \ddots & \ddots \end{pmatrix}, \quad \text{Dom} \ A = \text{span}\{e_n \mid n \in \mathbb{N}_0\}. \]
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Thus, to a given sequence of moments \( \{s_n\}_{n=0}^{\infty} \), we can find real \( \{b_n\}_{n=0}^{\infty} \) and positive \( \{a_n\}_{n=0}^{\infty} \) which give rise to the operator \( A \) and the spectral measures of its self-adjoint realization yield (some) solutions to the moment problem.
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There are explicit formulas for the \( b_n \)'s and \( a_n \)'s in terms of the determinants of the \( s_n \)'s.
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Consequently, we obtained the following correspondences:

- moment sequence \( \leftrightarrow \) Jacobi matrix
- \( \downarrow \) \( \uparrow \)
- Orthogonal Polynomials \( \leftrightarrow \) three-term recurrence
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Sufficient conditions for determinacy - moment sequence

It is desirable to be able to decide whether the moment problem is determinate (or indeterminate) just by looking at the moment sequence \( \{s_n\}_{n=0}^{\infty} \), or the Jacobi matrix (seq. \( \{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty} \)), or orthogonal polynomials \( \{P_n\}_{n=0}^{\infty} \).

Theorem (Carleman, 1922, 1926):

1) \( \sum_{n=1}^{\infty} \frac{n}{s_n} = \infty \) or 2) \( \sum_{n=1}^{\infty} \frac{1}{a_n} = \infty \)

then the Hamburger moment problem is determinate.

If \( \sum_{n=1}^{\infty} \frac{n}{s_n} = \infty \)

then both Hamburger and Stieltjes moment problems are determinate.

Hence, e.g., if \( \{a_n\}_{n=0}^{\infty} \) is bounded or there are \( R, C > 0 \) such that \( \left| s_n \right| \leq CR^n n! \), for all \( n \) sufficiently large, we have determinate Hamburger m.p. If \( \left| s_n \right| \leq CR^n (2^n)^n \), for all \( n \) sufficiently large, we have determinate Stieltjes m.p.
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If

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Theorem (Carleman, 1922, 1926):

If
\begin{align*}
1) \quad \sum_{n=1}^{\infty} \frac{1}{2n \sqrt{|s_{2n}|}} = \infty \quad \text{or} \quad 2) \quad \sum_{n=1}^{\infty} \frac{1}{a_n} = \infty
\end{align*}

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Theorem (Chihara, 1989):

Let

\[ \lim_{n \to \infty} b_n = \infty \quad \text{and} \quad \lim_{n \to \infty} \frac{a_n^2}{b_n b_{n+1}} = L < \frac{1}{4}. \]

then the Hamburger moment problem is determinate if

\[ \liminf_{n \to \infty} \sqrt[n]{b_n} < \frac{1 + \sqrt{1 - 4L}}{1 - \sqrt{1 - 4L}} \]

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- Chihara uses totally different approach to the problem - concept of chain sequences.
Recall \( \{P_n\}_{n=0}^{\infty} \) are determined by the three-term recurrence

\[
x P_n(x) = a_n P_{n+1}(x) + b_n P_n(x) + a_{n-1} P_{n-1}(x)
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with initial settings \( P_0(x) = 1 \) and \( P_1(x) = \frac{1}{b_0} (x - a_0) \).
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Let us denote by \( \{Q_n\}_{n=0}^{\infty} \) a polynomial sequence that solve the same recurrence as \( \{P_n\}_{n=0}^{\infty} \) with initial conditions \( Q_0(x) = 0 \) and \( Q_1(x) = \frac{1}{b_0} \).
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These two polynomial sequences are linearly independent and any solution of the three-term recurrence is a linear combination of them.

Theorem (Hamburger, 1920-21):
The Hamburger moment problem is determinate if and only if

$$\sum_{n=0}^\infty (P_{2n}(0) + Q_{2n}(0)) = \infty.$$
Sufficient conditions for determinacy

Recall \( \{ P_n \}_{n=0}^{\infty} \) are determined by the three-term recurrence

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The Hamburger moment problem is determinate if and only if

\[
\sum_{n=0}^{\infty} \left( P_n^2(0) + Q_n^2(0) \right) = \infty.
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Recall $\{P_n\}_{n=0}^\infty$ are determined by the three-term recurrence

$$x P_n(x) = a_n P_{n+1}(x) + b_n P_n(x) + a_{n-1} P_{n-1}(x)$$

with initial settings $P_0(x) = 1$ and $P_1(x) = \frac{1}{b_0} (x - a_0)$.

Let us denote by $\{Q_n\}_{n=0}^\infty$ a polynomial sequence that solve the same recurrence as $\{P_n\}_{n=0}^\infty$ with initial conditions $Q_0(x) = 0$ and $Q_1(x) = \frac{1}{b_0}$.

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Actually, one can write some \( x \in \mathbb{R} \) instead of zero in the condition.

It is even necessary and sufficient that there exists a \( z \in \mathbb{C} \setminus \mathbb{R} \) such that both \( \{ P_n(z) \}_{n=0}^{\infty} \) and \( \{ Q_n(z) \}_{n=0}^{\infty} \) does not belong to \( \ell^2(\mathbb{Z}_+) \).
Contents

1. Motivation
2. What the moment problem is?
3. Existence and uniqueness of the solution - operator approach
4. Jacobi matrix and Orthogonal Polynomials
5. Sufficient conditions for determinacy
6. The set of solutions of indeterminate moment problem
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A function $\phi$ is called Pick (or Nevanlinna–Pick or Herglotz–Nevanlinna) function if it is holomorphic in $\mathbb{C}_+ := \{z \in \mathbb{C} \mid \Im z > 0\}$ and $\Im \phi(z) \geq 0$ for $z \in \mathbb{C}_+$. 

Denote the set of Pick functions by $\mathcal{P}$. $\mathcal{P} \cup \{\infty\}$ denotes the one-point compactification of $\mathcal{P}$ ($\mathcal{P}$ inherits the topology of holomorphic functions on $\mathbb{C} \setminus \mathbb{R}$).

Nevanlinna, 1922

The solutions of the Hamburger moment problem in the indeterminate case are parametrized via homeomorphism $\phi \mapsto \mu_{\phi}$ of $\mathcal{P} \cup \{\infty\}$ onto $\mathcal{M}_H$ given by

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\[ \mu \text{ is an extreme point in } \mathcal{M}_H \text{ if and only if polynomials } C[x] \text{ are dense in } L^1(\mathbb{R}, \mu), \text{ [Naimark, 1946].} \]

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Note first that, for $k \in \mathbb{Z}_+$,

$$\int_0^\infty u^k u^{-\ln u} \sin(2\pi \ln u) du = 0.$$  

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  is in $L^2(d\mu_\vartheta)$ and it is orthogonal to all polynomials.

- Hence polynomials are not dense in $L^2(d\mu_\vartheta)$. This is a typical situation for solutions of indeterminate moment problems which are not N-extremal.
In some sense, to solve indeterminate Hamburger moment problem means to find the Nevanlinna functions $A, B, C, \text{ and } D$ (in particular $B$ and $D$).
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- In some sense, to solve indeterminate Hamburger moment problem means to find the Nevanlinna functions $A, B, C,$ and $D$ (in particular $B$ and $D$).
- They can be computed by using orthogonal polynomials,

$$A(z) = z \sum_{k=0}^{\infty} Q_k(0) Q_k(z), \quad C(z) = 1 + z \sum_{k=0}^{\infty} P_k(0) Q_k(z)$$

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More on $A, B, C, D$:

- $A, B, C, D$ are entire functions of order $\leq 1$, if the order is 1, the exponential type is 0 [Riesz, 1923]
- $A, B, C, D$ have the same order, type and Phragmén-Lindenlöf indicator function [Berg and Pedersen, 1994]
Important solutions 1/2

- If \( \phi(z) = t \in \mathbb{R} \cup \{\infty\} \) then \( \phi \in \mathcal{P} \cup \{\infty\} \) and \( \mu_t \) is a discrete measure of the form

\[
\mu_t = \sum_{x \in \Lambda_t} \rho(x)\delta(x).
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N-extremal solutions are indeed extreme points in $\mathcal{M}_H$ - but not the only ones.
If we set

$$
\phi(z) = \begin{cases} 
\beta + i\gamma, & \Im z > 0, \\
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for $\beta \in \mathbb{R}$ and $\gamma > 0$, then $\phi \in \mathcal{P}$ and $\mu_{\beta,\gamma}$ is absolutely continuous with density

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The solution \( \mu_{0,1} \) is the one that maximizes certain entropy integral, (see Krein’s condition). More general and additional information are provided in [Gabardo, 1992].
Nevanlinna parametrization in the case of Stieltjes moment problem

Suppose $\{s_n\}_{n=0}^{\infty}$ is a sequence of Stieltjes moments such that the moment problem is indeterminate in the sense of Hamburger.

The quantity $\alpha \leq 0$ plays an important role and can be obtained as the limit $\alpha = \lim_{n \to \infty} \frac{P_n(0)}{Q_n(0)}$.

The moment problem is determinate in the sense of Stieltjes if and only if $\alpha = 0$.

The only $\cal N$-extremal solutions supported within $[0, \infty)$ are $\mu_t$ with $\alpha \leq t \leq 0$.

For the indeterminate Stieltjes moment problem there is a slightly more elegant way how to describe $\cal M_S$ known as Krein parametrization, [Krein, 1967].
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- To describe \( \mathcal{M}_S \) one can still use the Nevanlinna parametrization.

\[ \alpha \leq \phi(x) \leq 0 \quad \text{for} \quad x < 0, \quad \text{[Pedersen, 1997]} \]

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Just restrict oneself to consider only the Pick functions \( \phi \) which have an analytic continuation to \( \mathbb{C} \setminus [0, \infty) \) such that \( \alpha \leq \phi(x) \leq 0 \) for \( x < 0 \), [Pedersen, 1997]
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- Suppose \( \{s_n\}_{n=0}^{\infty} \) is a sequence of Stieltjes moments such that the moment problem is indeterminate in the sense of Hamburger.
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For the indeterminate Stieljes moment problem there is a slightly more elegant way how to describe \( \mathcal{M}_S \) known as *Krein parametrization*, [Krein, 1967].
References:


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Thank you!