One-parameter generalization of some classes of orthogonal polynomials

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Outline

Definition

- A sequence of polynomials $\{P_n\}_{n=0}^{\infty}$ with real coefficients and P_n of degree n, for which there exists positive Borel measure μ on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} P_m(x)P_n(x)d\mu(x) = c_n\delta_{mn}, \quad m, n \in \mathbb{Z}_+,$$

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- Basic references:

N. I. Akhiezer: *The Classical Moment Problem and Some Related Questions in Analysis*, (Oliver & Boyd, Edinburgh, 1965).

T. S. Chihara: *An Introduction to Orthogonal Polynomials*, (Gordon and Breach, Science Publishers, Inc., New York, 1978).

M. E. H. Ismail *Classical and Quantum Orthogonal Polynomials in One Variable*, (Cambridge Univ. Press., Cambridge, 2005).

Numerical Analysis:

Applications of OPs in Mathematics 1/2

Numerical Analysis:

- *Approximation theory:* Monic Chebyshev polynomials $\tilde{T}_n$ possess "min-max" property on $[-1, 1]$:

$$\tilde{T}_n = \arg \min_{P \in \mathbb{P}_n} \max_{x \in [-1,1]} |P(x)|$$

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- *Differential equations:* Linear ordinary differential equations can be solved directly in series of Chebyshev polynomials (or other OPs) by a method originated by [Clenshaw, 1957]. This process has been generalized to spectral methods for solving partial differential equations, [Mason and Handscomb, 2003].

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Other Applications:

- *Integrable systems:* Toda equation provides important model of a completely integrable system. A wide class of exact solutions of the Toda equation can be expressed in terms of various special functions, and in particular OPs, [Nakamura, 1996]. For instance,

$$V_n(x) = 2nH_{n-1}(x)H_{n+1}(x)/H_n^2(x),$$

  where H_n are Hermite OPs, satisfies Toda equation

$$\frac{d^2}{dx^2} \log V_n(x) = V_{n+1}(x) - 2V_n(x) + V_{n-1}(x).$$

Applications of OPs in Mathematics 2/2

- *Complex function theory:* The Askey-Gasper inequality for Jacobi OPs

$$\sum_{k=0}^{n} P_k^{(\alpha,0)}(x) \geq 0, \qquad (x \in [-1,1], \alpha > -1, n \in \mathbb{Z}_+)$$

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- *Coding Theory:* Application of Krawtchouk and *q*-Racah OPs, [Bannai, 1990].

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Other physical applications:

- *Electrostatics models:* For interpretations of zeros of OPs as equilibrium positions of charges in electrostatic problems (assuming logarithmic interaction), see [Ismail, 2000].

  More precisely, put at 1 and -1 two positive charges p and q, and with these fixed charges put n positive unit charges on $(-1,1)$ at the points $x_1,\ldots,x_n$. The mutual energy of all these charges is

$$U(x_1,\ldots,x_n) = p\sum_{i=1}^{n}\log\frac{1}{|1-x_i|} + q\sum_{i=1}^{n}\log\frac{1}{|1+x_i|} + \sum_{i<j}\log\frac{1}{|x_i-x_j|}$$

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- *Statistical mechanics:* Explicitly solvable models, [Baxter, 1981-82].

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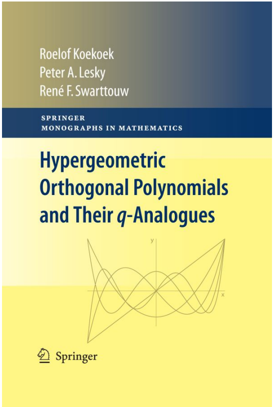

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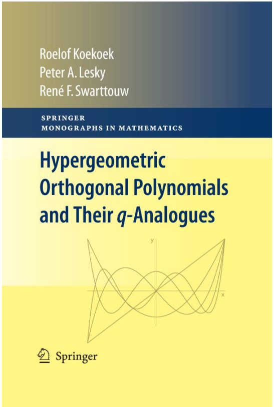

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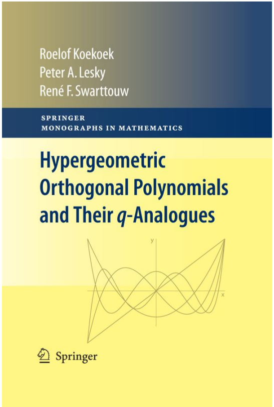

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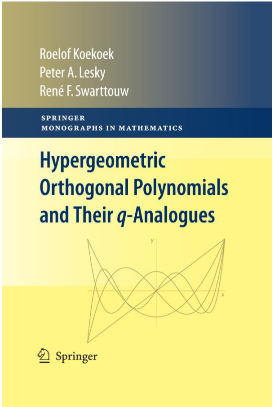
- Available for free on the web: http://aw.twi.tudelft.nl/∼koekoek/askey.html.



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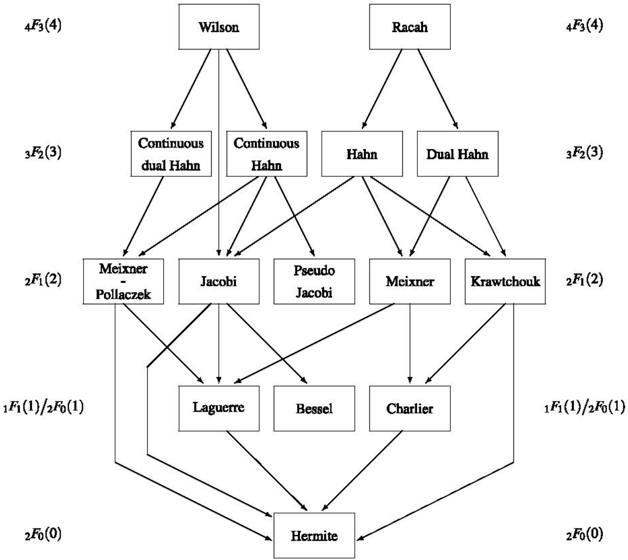

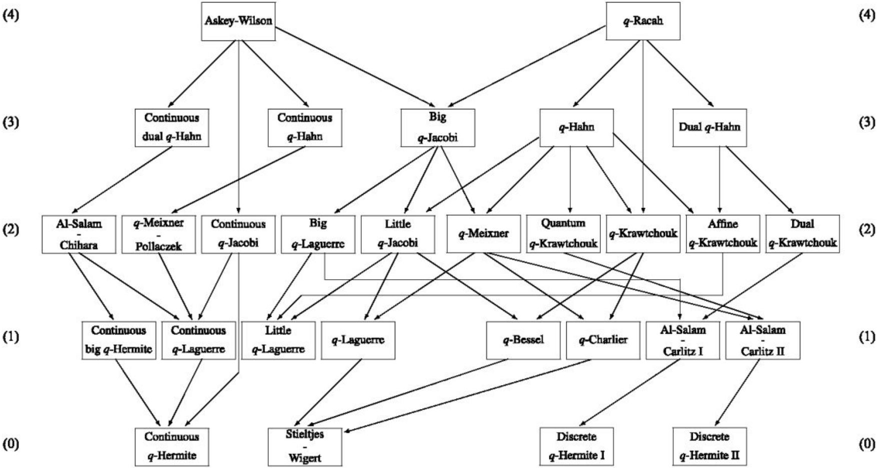
(4)
Askey-Wilson
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Continuous dual q-Hahn
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(2)
Al-Salam - Chihara
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- OPs can be defined by several equivalent ways. One possibility is the recursive definition via three-term recurrence rule

$$P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x),$$

with initial conditions $P_{-1}(x) = 0$ and $P_0(x) = 1$, where $c_n \in \mathbb{R}$ and $\lambda_n > 0$ (Favard's Theorem, [Chihara, Thm. 4.4]).

Basic characteristics of OPs - illustrated on Hermite OPs

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- Expression of OPs in terms of **special functions**:

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- **Asymptotic formulas** for large n:

$$e^{-\frac{x^2}{2}} H_n(x) \sim \frac{2^n}{\sqrt{\pi}} \Gamma\left(\frac{n+1}{2}\right) \cos\left(x\sqrt{2n} - n\frac{\pi}{2}\right).$$

- **Differential equation**:

$$y''(x) - 2xy'(x) + 2ny(x) = 0, \quad y(x) = H_n(x).$$

Basic characteristics of OPs - continuation

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- **Structure relations**:
  1. forward-shift operator,

$$\frac{d}{dx} H_n(x) = 2nH_{n-1}(x),$$

  2. backward-shift operator,

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- **OG relations**:

$$\int_{-\infty}^{\infty} H_m(x) H_n(x) e^{-x^2} dx = \sqrt{\pi} 2^n n! \delta_{mn}.$$

- Define polynomials $P_n(\alpha, \beta; x)$ recursively as solution of recurrence

$$u_{n+1} = (x - n - \beta)u_n - (\alpha + n\beta)u_{n-1}$$

with initial setting $P_{-1}(\alpha, \beta; x) = 0$ and $P_0(\alpha, \beta; x) = 1$.

Generalized Charlier OPs - definition & special function expression

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$$\left. -\beta^{n+1} \frac{\Gamma\left(n+1+\frac{\alpha}{\beta}\right)\Gamma(x-n)}{\Gamma\left(\frac{\alpha}{\beta}\right)\Gamma(x+2)} {}_1F_1\left(1 - \frac{\alpha}{\beta}; x+2; \beta\right) {}_1F_1\left(-\frac{\alpha}{\beta} - x; -x + n + 1; \beta\right) \right].$$

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$$-\beta^{n+1} \frac{\Gamma\left(n + 1 + \frac{\alpha}{\beta}\right) \Gamma(x - n)}{\Gamma\left(\frac{\alpha}{\beta}\right) \Gamma(x+2)} {}_1F_1\left(1 - \frac{\alpha}{\beta}; x + 2; \beta\right) {}_1F_1\left(-\frac{\alpha}{\beta} - x; -x + n + 1; \beta\right) \Bigg].$$

- Especially, for $\alpha = 0$, one has

$$C_n^{(\beta)}(x) = \frac{\Gamma(x+1)}{\Gamma(x+1-n)} \, {}_1F_1(-n; x - n + 1; \beta)\,.$$

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$$\sum_{n=0}^{\infty} \frac{P_n(\alpha, \beta; x)}{\Gamma(n + 1 + \alpha/\beta)} w^n = \frac{e^{-\beta w} w^{-\alpha/\beta} (1 + w)^{x+\alpha/\beta}}{\Gamma(\alpha/\beta)} \int_0^w e^{\beta t} t^{-1+\alpha/\beta} (1 + t)^{-1-x-\alpha/\beta} dt, \quad |w| < 1.$$

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- For $\alpha/\beta<0$ the generating function has **not** been found.

Generalized Charlier OPs - structure relation

- Structure relations: only “forward shift”,

$$P_n(\alpha,\beta;x+1)-P_n(\alpha,\beta;x)=\left(n+\frac{\alpha}{\beta}\right)P_{n-1}(\alpha,\beta;x)-\frac{\alpha}{\beta}P_{n-1}(\alpha+\beta,\beta;x).$$

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- The backward shift formula in the Charlier OPs case reads

$$\beta C_n^{(\beta)}(x)-xC_n^{(\beta)}(x-1)=-C_{n+1}^{(\beta)}(x).$$

Let $\beta > 0$ and $\alpha + \beta > 0$. Then it holds:

- Polynomials $P_n(\alpha, \beta; x)$ satisfy the orthogonality relation

$$\int_{\mathbb{R}} P_n(\alpha, \beta; x) P_m(\alpha, \beta; x) d\mu(x) = \beta^n \frac{\Gamma\left(\frac{\alpha}{\beta} + n + 1\right)}{\Gamma\left(\frac{\alpha}{\beta} + 1\right)} \delta_{mn}, \quad m, n \in \mathbb{Z}_+.$$

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- $d\mu$ is a purely discrete probability measure supported by infinite set with no finite cluster point given by

$$\operatorname{supp}(d\mu) = \left\{ x \in \mathbb{R} : {}_1\tilde{F}_1\left(-\frac{\alpha}{\beta} - x; -x; \beta\right) = 0 \right\}.$$

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Generalized Charlier OPs - orthogonality

- By sending $\alpha \to 0$, one reprove Charlier polynomials are orthogonal with respect to Poisson probability distribution,
$$\sum_{k=0}^{\infty} \frac{e^{-\beta}\beta^k}{k!} C_m^{(\beta)}(k) C_n^{(\beta)}(k) = \beta^n n! \delta_{mn},$$
for $\beta > 0$.

- Define polynomials $U_n(a, \delta; q, x)$ recursively as solution of recurrence

$$v_{n+1} = \left(x - (a+1)q^n\right) v_n + aq^{n+\delta-1}(1 - q^{n-\delta})v_{n-1},$$

with initial setting $U_{-1}(a, \delta; q, x) = 0$ and $U_0(a, \delta; q, x) = 1$.

Generalized Al-Salam-Carlitz I - definition & special function expression

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$$U_n(a,\delta;q,x) = \frac{1}{(ax^{-1}q^{\delta};q)_\infty}\Big[x^n(x^{-1};q)_n\,{}_1\phi_1\left(x^{-1}q^{\delta};x^{-1};q,ax^{-1}\right)\,{}_1\phi_1\left(q^{\delta-n};q^{1-n}x;q,aq\right)$$
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Generalized Al-Salam-Carlitz I OPs - generating function & asymptotics

- Asymptotic formula for $U_n(a,\delta;q,x)$ with $x \neq 0$, for $n \to \infty$:

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$$\sum_{n=0}^{\infty} \frac{U_n(a,\delta;q,x)}{(q^{1-\delta};q)_n} t^n = (1-q^{-\delta}) \sum_{k=0}^{\infty} \frac{(aq^{\delta}t;q)_k (q^{\delta}t;q)_k}{(xt;q)_{k+1}} q^{-k\delta}, \quad |xt| < 1,$$

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Generalized Al-Salam-Carlitz OPs - structure relation

- Structure relations: only “forward shift”,

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- By applying limit $\alpha \to 0$ in the last formula, one gets the forward shift formula for Al-Salam-Carlitz I,

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- The backward shift formula in the Al-Salam-Carlitz I OPs case reads

$$aU_n^{(a)}(x;q) - (1-x)(a-x)U_n^{(a)}(q^{-1}x;q) = -xq^{-n}U_{n+1}^{(a)}(x;q).$$

Generalized Al-Salam-Carlitz I OPs - orthogonality

Let $a, \delta < 0$ then it holds:

- Polynomials $U_n(a, \delta; q, x)$ satisfy the orthogonality relation

$$\int_{\mathbb{R}} U_m(a, \delta; q, x) U_n(a, \delta; q, x) d\mu(x) = (-a)^n q^{n\delta + n(n-1)/2} (q^{-\delta}; q)_{n+1} \delta_{mn}.$$

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- $d\mu$ is a purely discrete positive measure supported by infinite set, with 0 the only cluster point, given by

$$\operatorname{supp}(d\mu) = \left\{ x^{-1} \in \mathbb{R} : {}_2\phi_1\left(axq^{\delta}, xq^{\delta}; 0; q, q^{-\delta}\right) = 0 \right\} \cup \{0\},$$

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- Step function $\mu(x)$ has jumps at $x \in \mathrm{supp}(d\mu) \setminus \{0\}$ of magnitude

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- Explicit formulas and support of the measure of orthogonality for the generalization are expressed in terms of *(ir)regular Coulomb wave functions*.

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- This case is extremely interesting. The analysis of basic characteristics is much more difficult then in previous cases.
- Especially, concerning the measure of orthogonality, if $q \geq \tan^2(\sigma/2)$ then there is only one OG measure. However, if $q < \tan^2(\sigma/2)$ then there are infinitely many measures of orthogonality (cf. indeterminate moment problem).

Thank you, and enjoy Beskydy!