On certain function with applications concerning tridiagonal matrices

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Contents

1. Function $\mathfrak{F}$ and its fundamental properties
2. Application in the spectral analysis of Jacobi operators
3. Application in the theory of orthogonal polynomial

Definition

Let me define $\mathfrak{F} : D \to \mathbb{C}$ by relation

$$\mathfrak{F}(x) = 1 + \sum_{m=1}^{\infty}(-1)^m \sum_{k_1=1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \dots \sum_{k_m=k_{m-1}+2}^{\infty} x_{k_1} x_{k_1+1} x_{k_2} x_{k_2+1} \dots x_{k_m} x_{k_m+1},$$

where

$$D = \left\{ \{x_k\}_{k=1}^{\infty} \subset \mathbb{C};\ \sum_{k=1}^{\infty} |x_k x_{k+1}| < \infty \right\}.$$

For a finite number of complex variables let me identify $\mathfrak{F}(x_1, x_2, \dots, x_n)$ with $\mathfrak{F}(x)$ where $x = (x_1, x_2, \dots, x_n, 0, 0, 0, \dots)$.

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- Note the function $\mathfrak{F}$ is indeed well defined on the domain D since one has the estimate
$$|\mathfrak{F}(x)| \leq \exp\left(\sum_{k=1}^{\infty} |x_k x_{k+1}| \right).$$
- Note D is not a linear space. One has, however, $\ell^2(\mathbb{N}) \subset D$.

Fundamental properties of $\mathfrak{F}$

- For all $x \in D$, one has

$$\mathfrak{F}(x) = \mathfrak{F}(Tx) - x_1 x_2 \, \mathfrak{F}(T^2 x)$$

where T is the shift operator acting on the space of complex sequences as

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- This is a particular case of the more general formula:

$$\mathfrak{F}(x) = \mathfrak{F}(x_1, \ldots, x_k) \, \mathfrak{F}(T^k x) - \mathfrak{F}(x_1, \ldots, x_{k-1}) x_k x_{k+1} \mathfrak{F}(T^{k+1} x)$$

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- Equivalent definition for $\mathfrak{F}(x_1, x_2, \dots, x_n)$ is:

$$\mathfrak{F}(x_1, x_2, \dots, x_n) = \det X_n = \det \begin{pmatrix} 1 & x_1 & & & \\ x_2 & 1 & x_2 & & \\ & \ddots & \ddots & \ddots & \\ & & x_{n-1} & 1 & x_{n-1} \\ & & & x_n & 1 \end{pmatrix}.$$

For $x \in D$, we have $\mathfrak{F}(x) = \lim_{n \to \infty} \det X_n$.

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1 Function $\mathfrak{F}$ and its fundamental properties

2 **Application in the spectral analysis of Jacobi operators**

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Characteristic function of Jacobi matrix

- Let us denote

$$J = \begin{pmatrix} \lambda_1 & w_1 & & & \\ w_1 & \lambda_2 & w_2 & & \\ & w_2 & \lambda_3 & w_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

where $\lambda_n \in \mathbb{R}$ and $w_n \in \mathbb{R} \setminus \{0\}$.

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where $\lambda_n \in \mathbb{R}$ and $w_n \in \mathbb{R} \setminus \{0\}$.
- Let J_N denotes the $N \times N$ principal submatrix of J.
- The characteristic function of J_N can be written in the following form
$$\det(J_N - z) = \left(\prod_{k=1}^{N}(\lambda_k - z)\right) \mathfrak{F}\left(\frac{\gamma_1^2}{\lambda_1 - z}, \frac{\gamma_2^2}{\lambda_2 - z}, \dots, \frac{\gamma_N^2}{\lambda_N - z}\right),$$
where $\{\gamma_k\}_{k=1}^{N}$ is any sequence satisfying the recurrence $\gamma_k\gamma_{k+1} = w_k$, for $k \geq 1$.

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where $\{\gamma_k\}_{k=1}^{N}$ is any sequence satisfying the recurrence $\gamma_k \gamma_{k+1} = w_k$, for $k \geq 1$.
- By extracting the term with $\mathfrak{F}$ and sending $N \to \infty$ one arrives at the function

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is well defined if there exists at least one $z_0 \in \mathbb{C} \setminus \overline{\operatorname{Ran} \lambda}$ such that

$$\sum_{n=1}^{\infty} \frac{w_n^2}{|(\lambda_n - z_0)(\lambda_{n+1} - z_0)|} < \infty.$$

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- Function F_J is analytic on $\mathbb{C} \setminus \overline{\mathrm{Ran}\,\lambda}$ and we call it the characteristic function of the Jacobi matrix J.

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Moreover, if $z \in \mathbb{C} \setminus \overline{\operatorname{Ran} \lambda}$ is an eigenvalue of J then the vector $\xi(z) = (\xi_1(z), \xi_2(z), \dots)$ where

$$\xi_n(z) = \left(\prod_{k=1}^{n} \frac{w_{k-1}}{z - \lambda_k} \right) \mathfrak{F}\left(\left\{ \frac{\gamma_k^2}{\lambda_k - z} \right\}_{k=n+1}^{\infty} \right), \quad (w_0 := 1),$$

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Furthermore, for the Green function $G_{ij}(z) = (e_i, (J-z)^{-1} e_j)$ we have

$$G_{ij}(z) = -\frac{1}{w_M} \prod_{l=m}^{M} \left(\frac{w_l}{z - \lambda_l} \right) \frac{\mathfrak{F}\left(\left\{ \frac{\gamma_l^2}{\lambda_l - z} \right\}_{l=1}^{m-1} \right) \mathfrak{F}\left(\left\{ \frac{\gamma_l^2}{\lambda_l - z} \right\}_{l=M+1}^{\infty} \right)}{\mathfrak{F}\left(\left\{ \frac{\gamma_l^2}{\lambda_l - z} \right\}_{l=1}^{\infty} \right)}$$

where $z \notin \operatorname{spec}(J)$, $m := \min(i,j)$, and $M := \max(i,j)$.

Example

- Set $\lambda_n = n$ and $w_n = w \in \mathbb{R} \setminus \{0\}$, for $n \in \mathbb{N}$. Thus, in this case,

$$J = \begin{pmatrix} 1 & w & & & \\ w & 2 & w & & \\ & w & 3 & w & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

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- One has

$$\mathfrak{F}\left(\left\{\frac{\gamma_k^2}{k-z}\right\}_{k=r+1}^{\infty}\right) = \mathfrak{F}\left(\left\{\frac{w}{k-z}\right\}_{k=r+1}^{\infty}\right) = w^{z-r}\Gamma(1+r-z)J_{r-z}(2w)$$

for $r \in \mathbb{Z}_+$.

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- The general theorem tells us

$$\operatorname{spec}(J) = \{z \in \mathbb{R} \mid J_{-z}(2w) = 0\}$$

and components of corresponding eigenvectors $v(z)$ can be chosen as

$$v_k(z) = (-1)^k J_{k-z}(2w), \ k \in \mathbb{N}.$$

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- For $\lambda_n \in \mathbb{R}$ and $w_n > 0$, consider the symmetric second order difference equations

$$w_{n-1}y_{n-1}(x) + \lambda_n y_n(x) + w_n y_{n+1}(x) = xy_n(x), \quad n = 1, 2, \dots \quad (w_0 := -1).$$

OPs of the first kind $P_n(x)$ are the solution satisfying initial conditions $P_0(x) = 0$, $P_1(x) = 1$, while OPs of the second kind $Q_n(x)$ satisfy the same recurrence starting with the initial conditions $Q_0(x) = 1$, $Q_1(x) = 0$.

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- OPs are related to $\mathfrak{F}$ through identities

$$P_{n+1}(z) = \prod_{k=1}^{n} \left(\frac{z - \lambda_k}{w_k} \right) \mathfrak{F} \left(\left\{ \frac{\gamma_l^2}{\lambda_l - z} \right\}_{l=1}^{n} \right), \quad n = 0, 1 \ldots,$$

$$Q_{n+1}(z) = \frac{1}{w_1} \prod_{k=2}^{n} \left(\frac{z - \lambda_k}{w_k} \right) \mathfrak{F} \left(\left\{ \frac{\gamma_l^2}{\lambda_l - z} \right\}_{l=2}^{n} \right), \quad n = 0, 1 \ldots$$

where $\{\gamma_n\}$ can be defined recursively by $\gamma_k \gamma_{k+1} = w_k$.

Limit formula for OPs

Proposition

If $\sum_{k\geq 0}\left|\frac{w_k^2}{(z-\lambda_k)(z-\lambda_{k+1})}\right| < \infty$, for some $z \in \mathbb{C}$, then for all $z \in \mathbb{C}\setminus\mathbb{R}$ we have

$$\left(\prod_{k=1}^{n-1}\frac{w_k}{z-\lambda_k}\right)P_n(z) \quad \underset{n\to\infty}{\longrightarrow} \quad \mathfrak{F}\left(\left\{\frac{\gamma_k^2}{\lambda_k - z}\right\}_{k=1}^{\infty}\right).$$

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Typical example: By setting $\lambda_n = 0$ and $w_n = [4(n+\nu-1)(n+\nu)]^{-1/2}$, polynomials

$$P_{n+1}(x) = (2x)^n\sqrt{\frac{\nu}{\nu+n}}\frac{\Gamma(n+\nu)}{\Gamma(\nu)}\,\mathfrak{F}\left(\left\{\frac{1}{2x(\nu+k-1)}\right\}_{k=1}^{n}\right)$$

are related to Lommel polynomials $R_{n,\nu}(x)$:

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The above limit relation yields the *Hurwitz's asymptotic formula for Lommel polynomials*

$$\lim_{n\to\infty}\frac{x^n}{2^n\Gamma(\nu+n)}R_{n,\nu}(x)=\left(\frac{x}{2}\right)^{-\nu+1}J_{\nu-1}(x).$$

Constructing measure of orthogonality

Theorem (essentially due to Markov)

Let the Hmp corresponding to P_n be determinate. Then

$$\lim_{n\to\infty} \frac{Q_n(z)}{P_n(z)} = \int_{\mathbb{R}} \frac{d\mu(x)}{z-x}, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

where μ is the measure of orthogonality for P_n.

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- Under certain assumption on λ_n and w_n, we can apply the limit formula for $P_n(x)$ (and similar formula for $Q_n(x)$):

$$\int_{\mathbb{R}}\frac{\mathrm{d}\mu(x)}{z-x}=\frac{\mathfrak{F}\left(\left\{\frac{\gamma_k^2}{\lambda_k-z}\right\}_{k=2}^{\infty}\right)}{(z-\lambda_1)\mathfrak{F}\left(\left\{\frac{\gamma_k^2}{\lambda_k-z}\right\}_{k=1}^{\infty}\right)},$$

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  for $z\notin\operatorname{supp}\mu$.

- Having the Stieltjes transform of μ we can, in principle, determine the measure of orthogonality μ by using the Stieltjes-Perron inversion formula.

Measure of orthogonality - special case

- Assume $\lambda_n = 0$ and $\{w_n\} \in \ell^2$ then the $\operatorname{supp}\mu$ is a denumerable set of points with 0 the only accumulation point and we have

$$\int_{\mathbb{R}} \frac{\mathrm{d}\mu(x)}{z-x} = \frac{\mathfrak{F}\left(\left\{z^{-1}\gamma_k^2\right\}_{k=2}^{\infty}\right)}{z\,\mathfrak{F}\left(\left\{z^{-1}\gamma_k^2\right\}_{k=1}^{\infty}\right)}, \quad z \notin \operatorname{supp}\mu.$$

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- If we denote $\operatorname{supp}(\mu) \setminus \{0\} = \{\mu_1, \mu_2, \dots\}$ then the last equality can be rewritten as the Mittag-Leffler expansion:

$$\frac{\Lambda_0}{z} + \sum_{k=1}^{\infty} \frac{\Lambda_k}{z-\mu_k} = \frac{\mathfrak{F}\left(\left\{z^{-1}\gamma_k^2\right\}_{k=2}^{\infty}\right)}{z\,\mathfrak{F}\left(\left\{z^{-1}\gamma_k^2\right\}_{k=1}^{\infty}\right)}.$$

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- From this expression, one extracts information about μ_k and Λ_k. Namely, we have

$$\operatorname{supp}(\mu)\setminus\{0\} = \{\mu_1, \mu_2, \dots\} = \left\{\frac{1}{x} \in \mathbb{R} \;:\; \mathfrak{F}\left(\left\{x\gamma_k^2\right\}_{k=1}^{\infty}\right) = 0\right\},$$

and

$$\Lambda_n = \mathfrak{F}\left(\left\{\mu_n^{-1}\gamma_k^2\right\}_{k=2}^{\infty}\right)\left(\frac{d}{dz}\bigg|_{z=\mu_n} z\,\mathfrak{F}\left(\left\{z^{-1}\gamma_k^2\right\}_{k=1}^{\infty}\right)\right)^{-1}.$$

Example

- Let us set $\lambda_n = 0$ and $w_n = 1/\sqrt{(\nu+n)(\nu+n+1)}$ then

$$\mathfrak{F}\left(\left\{z^{-1}\gamma_k^2\right\}_{k=1}^{\infty}\right) = \mathfrak{F}\left(\left\{\frac{1}{z(\nu+k)}\right\}_{k=1}^{\infty}\right) = \Gamma(\nu+1)\, z^{\nu} J_{\nu}(2/z)$$

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- Hence, by the general result, the measure of orthogonality for P_n is supported by the set

$$\left\{\frac{2}{\pm j_{k,\nu}}\right\}_{k=1}^{\infty} \text{ zeros of the function } z \mapsto z^{\nu} J_{\nu}(2/z).$$

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which can be further simplified to the well known relation

$$\sum_{k=1}^{\infty} j_{k,\nu}^{-2}\, R_{n,\nu+1}(\pm j_{k,\nu}) R_{m,\nu+1}(\pm j_{k,\nu}) = \frac{1}{2(n+\nu+1)}\,\delta_{mn}$$

where $R_{n,\nu}$ stands for the Lommel polynomial (the standard notation).

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- There is a close relation between $\mathfrak{F}$ and **continued fractions**.

References

1. F. Š. and P. Šťovíček: *On the Eigenvalue Problem for a Particular Class of Finite Jacobi Matrices*, **arXiv:1011.1241**.
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4. F. Š. and P. Šťovíček: *Orthogonal polynomials associated with Coulomb wave functions*, **arXiv:1403.8083**.

References

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Thank you!