Nevanlinna extremal measures for polynomials related to q-Fibonacci polynomials

František Štampach

Faculty of Information Technology, CTU in Prague

Symmetries of Discrete Systems and Processes

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Contents

- Let $0 < q < 1$, $r, s \in \mathbb{Z}_+$. Recall the basic hypergeometric function

$$ {}_r\phi_s \left[\begin{matrix} a_1, & a_2, & \dots & a_r \\ b_1, & b_2, & \dots & b_s \end{matrix} ; q, z \right] $$

is defined by the power series

$$ \sum_{n=0}^{\infty} \frac{(a_1;q)_n (a_2;q)_n \dots (a_r;q)_n}{(b_1;q)_n (b_2;q)_n \dots (b_s;q)_n} \frac{(-1)^{(s-r+1)n} q^{(s-r+1)n(n-1)/2}}{(q;q)_n} z^n $$

where $z, a_1, a_2, \dots, a_r \in \mathbb{C}$, $b_1, b_2, \dots, b_s \in \mathbb{C} \setminus q^{\mathbb{Z}_-}$ and

$$ (a;q)_0 = 1, \quad (a;q)_n = (1-a)(1-aq)\dots(1-aq^{n-1}) $$

is the q-Pochhammer symbol.

- Two commonly known q-analogues to exponential function are due to Jackson:

$$E_q(z) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q;q)_n} z^n \quad \text{and} \quad e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n}.$$

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- One easily verifies that

$$\lim_{q \to 1-} \mathcal{E}_q((1-q)z) = \exp(z).$$

q-trigonometric functions

- Let us introduce the couple of *q-sine* and *q-cosine* such that the q-version of Euler's identity

$$\mathcal{E}_q(iz) = \mathcal{C}_q(z) + iq^{1/4}\mathcal{S}_q(z)$$

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$$\mathcal{S}_q(z) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)}}{(q;q)_{2n+1}} z^{2n+1} \quad \text{and} \quad \mathcal{C}_q(z) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(q;q)_{2n}} z^{2n}.$$

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- Alternatively, functions $\mathcal{S}_q$ and $\mathcal{C}_q$ can be written as the ${}_1\phi_1$ function with the base q^2,

$$\mathcal{S}_q(z) = \frac{z}{1-q}\, {}_1\phi_1(0; q^3; q^2, q^2z^2) \quad \text{and} \quad \mathcal{C}_q(z) = {}_1\phi_1(0; q; q^2, qz^2).$$

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- Functions $\mathcal{S}_q$ and $\mathcal{C}_q$ possess many nice properties. Let us only mention that they can be expressed with the aid of the third Jackson (or Hahn-Exton) q-Bessel function. In addition, they form a couple of linearly independent solution to a second-order q-difference equation.

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- At last, let us define the corresponding q-analogue to the *hyperbolic sine* and *cosine*:

$$\mathcal{S}h_q(z) = -i\mathcal{S}_q(iz) \quad \text{and} \quad \mathcal{C}h_q(z) = \mathcal{C}_q(iz).$$

Proposition

For $u, v \in \mathbb{C}$, it holds

$$\mathcal{E}_q(u)\mathcal{E}_q(-v) = {}_3\phi_3\left[\begin{matrix}0, & u^{-1}vq^{1/2}, & uv^{-1}q^{1/2} \\ q^{1/2}, & -q^{1/2}, & -q\end{matrix}; q, uvq^{1/2}\right] + q^{1/4}\frac{u-v}{1-q}\,{}_3\phi_3\left[\begin{matrix}0, & u^{-1}vq, & uv^{-1}q \\ q^{3/2}, & -q^{3/2}, & -q\end{matrix}; q, uvq\right].$$

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Corollaries:

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$$\mathcal{C}_q(u)\mathcal{C}_q(v) + q^{1/2}\mathcal{S}_q(u)\mathcal{S}_q(v) = {}_3\phi_3\left[\begin{matrix} 0, & u^{-1}vq^{1/2}, & uv^{-1}q^{1/2} \\ q^{1/2}, & -q^{1/2}, & -q \end{matrix}; q, -uvq^{1/2}\right],$$

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By setting $u = q^{1/2}v$ in 1. one gets

$$\mathcal{C}_q(q^{1/2}v)\mathcal{C}_q(v) + q^{1/2}\mathcal{S}_q(q^{1/2}v)\mathcal{S}_q(v) = 1.$$

Contents

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$$\varphi_n(x;q) = \sum_{2k<n} \begin{bmatrix} n-k-1 \\ k \end{bmatrix}_q q^{k^2} x^{n-2k-1}$$

where $n \in \mathbb{Z}_+$ and

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$$\varphi_{n+1}(x;q) = x\varphi_n(x;q) + q^{n-1}\varphi_{n-1}(x;q), \quad n \in \mathbb{N},$$

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  with the initial conditions $\varphi_0(x;q) = 0$ and $\varphi_1(x;q) = 1$.
- Let us mention that $\varphi_n(1;q)$ are polynomials in q first considered by I. Schur (1917) in conjunction with his proof of Rogers-Ramanujan identities. They are referred as q-Fibonacci numbers F_n since clearly $\varphi_n(1;1) = F_n$.

Relation between $\mathcal{E}_q$ and q-Fibonacci polynomials

- For convenience, we replace q with q^{-1} from now.

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Proposition

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Corollary:

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Asymptotic behavior of q^{-1}-Fibonacci polynomials

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Proposition

For all $x \in \mathbb{C}$ and $0 < q < 1$, the limit relations

$$\lim_{n\to\infty} q^{n(n-1)} \varphi_{2n}(x; q^{-1}) = \mathcal{S}h_q(x) \quad \text{and} \quad \lim_{n\to\infty} q^{n^2} \varphi_{2n+1}(x; q^{-1}) = \mathcal{C}h_q(x)$$

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hold.

- Let us note the asymptotic behavior in case of q-Fibonacci polynomials with $0 < q < 1$ is particularly different:

$$\lim_{n\to\infty} x^{-n}\varphi_{n+1}(x;q) = {}_0\phi_1(;0;q,qx^{-2}).$$

Contents

Orthogonal polynomials associated with q-Fibonacci polynomials

- Recall polynomials $\varphi_n(x;q)$ are a solution of the second-order recurrence

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- If we put

$$T_n(x;q) = (-\mathrm{i})^n q^{-n/2}\varphi_{n+1}(\mathrm{i}q^{1/2}x;q), \quad n = -1, 0, 1, 2, \dots,$$

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then $\{T_n(x;q)\}$ fulfills the second-order difference equation

$$T_{n+1}(x;q) = xT_n(x;q) - q^{n-1}T_{n-1}(x;q), \quad n \in \mathbb{Z}_+,$$

with the initial conditions $T_{-1}(x;q) = 0$ and $T_0(x;q) = 1$.

- For $q > 0$ The Favard's theorem is applicable to the family $\{T_n(x;q)\}$. It tells us that there exists a positive Borel measure such that polynomials $\{T_n(x;q)\}$ are OG w.r.t. this measure.

- Recall polynomials $\varphi_n(x;q)$ are a solution of the second-order recurrence

$$\varphi_{n+1}(x;q) = x\varphi_n(x;q) + q^{n-1}\varphi_{n-1}(x;q), \quad n \in \mathbb{N},$$

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- For $q > 0$ The Favard's theorem is applicable to the family $\{T_n(x;q)\}$. It tells us that there exists a positive Borel measure such that polynomials $\{T_n(x;q)\}$ are OG w.r.t. this measure.
- In addition, it is not hard to show that
  1. the measure of OG is unique iff $0 < q \leq 1$ (determinate case of Hmp) and
  2. there infinitely many measures of OG iff $q > 1$ (indeterminate case of Hmp).

Measure of OG in determinate case

- The measure of OG in the case $0 < q < 1$ has been found by Al-Salam and Ismail (1983):

$$\sum_{j=1}^{\infty} \frac{\Phi_q(qz_j(q))}{z_j(q)\Phi_q'(z_j(q))} T_n(\pm z_j^{-1/2}) T_m(\pm z_j^{-1/2}) = -2q^{n(n-1)/2}\delta_{mn},$$

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$$T_n\left(z_j^{-1/2}(q); q\right) T_m\left(z_j^{-1/2}(q); q\right) + T_n\left(-z_j^{-1/2}(q); q\right) T_m\left(-z_j^{-1/2}(q); q\right),$$

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- The case $q = 1$ corresponds to Chebyshev polynomials of the second kind. Their measure of orthogonality is very well known.

Assume the indeterminate case. Recall the **Nevanlinna Theorem**:

- All measures of orthogonality μ_φ are in one-to-one correspondence with functions φ belonging to the one-point compactification of the space of Pick functions.

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- The correspondence is established by identifying the Stieltjes transform of the measure μ_φ,

$$\int_{\mathbb{R}} \frac{\mathrm{d}\mu_\varphi(x)}{z-x} = \frac{A(z)\varphi(z)-C(z)}{B(z)\varphi(z)-D(z)}\,, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

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- Four entire functions A, B, C, D are called Nevanlinna functions and they are determined by the leading term of the asymptotic expansion of corresponding OG polynomials of the first and second kind for large index.

Nevanlinna extremal measures

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- Hence

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- For the weight function ρ one has

$$\rho(x) = \frac{1}{B'(x)D(x) - B(x)D'(x)}.$$

- Since we have the limit relation for polynomials $\varphi_n(x;q^{-1})$, for $n \to \infty$, we can express functions A, B, C, D in terms of $\mathcal{S}_q$ and $\mathcal{C}_q$.

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Proposition

For $0 < q < 1$, Nevanlinna functions corresponding to OG polynomials $\{T_n(x;q^{-1})\}$ are as follows:

$$A(z) = q^{-1/2}D(q^{1/2}z) = \mathcal{S}_q(z) \quad \text{and} \quad C(z) = -B(q^{1/2}z) = \mathcal{C}_q(z).$$

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- These formulas are not new. They have already been obtain by Chen and Ismail (1998).

First application of the product formula - the reproducing kernel

- Recall the reproducing kernel for polynomials $T_n(x; q^{-1})$ is related with Nevanlinna functions B and D by formula
$$K(u,v) = \frac{B(u)D(v) - D(u)B(v)}{u - v}.$$

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The formula for the reproducing kernel for orthogonal polynomials $T_n(x;q^{-1})$ reads

$$K(u,v) = \frac{1}{1-q}\, {}_3\phi_3\left[\begin{matrix} 0, & u^{-1}vq, & uv^{-1}q \\ q^{3/2}, & -q^{3/2}, & -q \end{matrix}; q, -uv\right].$$

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- By applying the limit $v \to u$ in the last expression one finds the formula for the weight function ρ:

$$\frac{1}{\rho(u)} = B'(u)D(u) - D'(u)B(u) = \frac{1}{1-q}\, {}_3\phi_3\left[\begin{matrix} 0, & q, & q \\ q^{3/2}, & -q^{3/2}, & -q \end{matrix}; q, -u^2\right].$$

Second application of the product formula - support of N-extremal measure

- Recall N-extremal measure μ_t is supported by zeros of the function

$$z \mapsto B(z)t - D(z).$$

where $B(z) = -\mathcal{C}_q(q^{-1/2}z)$ and $D(z) = q^{1/2}\mathcal{S}_q(q^{-1/2}z)$.

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- By applying the suitable reparametrization

$$t = \frac{\mathcal{C}_q(u)}{\mathcal{S}_q(u)}$$

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- Applying the product formula once more, we get the final complete description of all N-extremal measures of orthogonality of polynomials $T_n(x;q^{-1})$ ...

Theorem

If $0<q<1$ and $u\in\mathbb{R}$, then the orthogonality relation for $T_n(x;q^{-1})$ reads

$$\sum_{k=1}^{\infty}\left({}_3\phi_3\left[\begin{matrix}0, & q, & q\\ q^{3/2}, & -q^{3/2}, & -q\end{matrix};q,-\lambda_k^2(u)\right]\right)^{-1}T_n(\lambda_k(u);q^{-1})T_m(\lambda_k(u);q^{-1})=\frac{q^{-n(n-1)/2}}{1-q}\delta_{mn}$$

where $\lambda_1(u),\lambda_2(u),\lambda_3(u),\dots$ stand for zeros of the function

$$z\mapsto {}_3\phi_3\left[\begin{matrix}0, & u^{-1}z, & uz^{-1}q\\ q^{1/2}, & -q^{1/2}, & -q\end{matrix};q,-uz\right].$$

Two particular orthogonality relations

- Let the sequences

$$0 < s_1(q) < s_2(q) < s_3(q) < \dots \quad \text{and} \quad 0 < c_1(q) < c_2(q) < c_3(q) < \dots$$

denote all positive zeros of $\mathcal{S}_q$ and $\mathcal{C}_q$, respectively.

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$$(1-q)T_n(0;q^{-1})T_m(0;q^{-1}) - \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{\mathcal{S}_q(qs_k(q))}{s_k(q)\mathcal{S}_q'(s_k(q))} T_n(q^{\frac{1}{2}}s_k(q);q^{-1})T_m(q^{\frac{1}{2}}s_k(q);q^{-1}) = q^{-n(n-1)/2}\delta_{mn}$$

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   where $s_{-k}(q) = -s_k(q)$.

2. $$-\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{\mathcal{C}_q(qc_k(q))}{c_k(q)\mathcal{C}_q'(c_k(q))} T_n(q^{\frac{1}{2}}c_k(q);q^{-1})T_m(q^{\frac{1}{2}}c_k(q);q^{-1}) = q^{-n(n-1)/2}\delta_{mn}$$

   where $c_{-k}(q) = -c_k(q)$.

Reference

- F. Š.: *Nevanlinna extremal measures for polynomials related to q^{-1}-Fibonacci polynomials*, arXiv:1505.00742.

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Thank you!