The characteristic function for infinite Jacobi matrices, the spectral zeta function, and solvable examples

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$$Je_n = w_{n-1}e_{n-1} + \lambda_n e_n + w_n e_{n+1} \quad (w_0 := 0)$$

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Introduction

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- Objective: Investigation of the spectrum of J when the diagonal sequence dominates the off-diagonal one in some sense.

Motivation – Reformulation of the problem

- For $z \in \mathbb{C}$ and $\lambda_n > 0$ define

$$A(z) := L^{-1/2}(UW + WU^* - z)L^{-1/2} = \begin{pmatrix} -\frac{z}{\lambda_1} & \frac{w_1}{\sqrt{\lambda_1\lambda_2}} & & & \\ \frac{w_1}{\sqrt{\lambda_1\lambda_2}} & -\frac{z}{\lambda_2} & \frac{w_2}{\sqrt{\lambda_2\lambda_3}} & & \\ & \frac{w_2}{\sqrt{\lambda_2\lambda_3}} & -\frac{z}{\lambda_3} & \frac{w_3}{\sqrt{\lambda_3\lambda_4}} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

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Main advantages are:

- $A(z)$ is Hilbert-Schmidt while J is unbounded;
- one can use function $z \mapsto \det_2(1 + A(z))$ which is well defined as an entire function.

Definition

Let me define $\mathfrak{F} : D \to \mathbb{C}$ by relation

$$\mathfrak{F}(x) = 1 + \sum_{m=1}^{\infty} (-1)^m \sum_{k_1=1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \dots \sum_{k_m=k_{m-1}+2}^{\infty} x_{k_1} x_{k_1+1} x_{k_2} x_{k_2+1} \dots x_{k_m} x_{k_m+1},$$

where

$$D = \left\{ \{x_k\}_{k=1}^{\infty} \subset \mathbb{C};\ \sum_{k=1}^{\infty} |x_k x_{k+1}| < \infty \right\}.$$

For a finite number of complex variables let me identify $\mathfrak{F}(x_1, x_2, \dots, x_n)$ with $\mathfrak{F}(x)$ where $x = (x_1, x_2, \dots, x_n, 0, 0, 0, \dots)$.

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- Note that the domain D is not a linear space. One has, however, $\ell^2(\mathbb{N}) \subset D$.

- For all $x \in D$ and $k = 1, 2, \ldots$ one has

$$\mathfrak{F}(x) = \mathfrak{F}(x_1, \ldots, x_k)\,\mathfrak{F}(T^k x) - \mathfrak{F}(x_1, \ldots, x_{k-1})x_k x_{k+1}\mathfrak{F}(T^{k+1} x)$$

where T denotes the shift operator from the left defined on the space of complex sequences:

$$T(\{x_k\}_{k=1}^{\infty}) = \{x_{k+1}\}_{k=1}^{\infty}.$$

Properties of $\mathfrak{F}$

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- Functions $\mathfrak{F}$ restricted on $\ell^2(\mathbb{N})$ is a continuous functional on $\ell^2(\mathbb{N})$. Further, for $x \in D$, it holds

$$\lim_{n\to\infty} \mathfrak{F}(x_1, x_2, \ldots, x_n) = \mathfrak{F}(x) \qquad \text{and} \qquad \lim_{n\to\infty} \mathfrak{F}(T^n x) = 1.$$

- For $\mathfrak{F}(x_1, x_2, \ldots, x_n)$ it holds:

$$\mathfrak{F}(x_1, x_2, \ldots, x_n) = \det \begin{pmatrix} 1 & x_1 & & & \\ x_2 & 1 & x_2 & & \\ & \ddots & \ddots & \ddots & \\ & & x_{n-1} & 1 & x_{n-1} \\ & & & x_n & 1 \end{pmatrix}.$$

Other properties of $\mathfrak{F}$

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- Equivalently we can define $\mathfrak{F}(x)$, for $x \in D$, as the limit

$$\mathfrak{F}(x) = \lim_{n\to\infty} \mathfrak{F}(x_1, x_2, \dots, x_n).$$

Function $\mathfrak{F}$ and continued fractions

- Function $\mathfrak{F}$ is related to a continued fraction. For a given $x \in D$ such that $\mathfrak{F}(x) \neq 0$, it holds

$$\frac{\mathfrak{F}(Tx)}{\mathfrak{F}(x)} = \cfrac{1}{1 - \cfrac{x_1 x_2}{1 - \cfrac{x_2 x_3}{1 - \cfrac{x_3 x_4}{1 - \dots}}}}.$$

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Remark: By using properties of the function $\mathfrak{F}$ we can show that with the continued fraction of the above form (S-fraction) is unambiguously associated a formal power series $f(x) \in \mathbb{C}[[x]]$ where $x = \{x_1, x_2, \dots\}$ [Zajta&Pandikow, 1975].

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$$f(x) = 1 + \sum_{\ell=1}^{\infty} \sum_{m \in \mathbb{N}^\ell} \beta(m) \prod_{j=1}^{\ell} (x_j x_{j+1})^{m_j}.$$

where, for $m \in \mathbb{N}^\ell$, we denote

$$\beta(m) = \prod_{j=1}^{\ell-1} \binom{m_j + m_{j+1} - 1}{m_{j+1}}.$$

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Various special functions are expressible in terms of $\mathfrak{F}$ applied to a suitable sequence, e.g.:

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- Hypergeometric Functions ${}_0F_1$, especially Bessel Functions,

$$\mathfrak{F}\left(\left\{\frac{w}{k+z}\right\}_{k=1}^{\infty}\right) = {}_0F_1(;z+1,-w^2) = \Gamma(1+z)\,w^{-z}J_z(2w)$$

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- q-Hypergeometric Functions ${}_0\phi_1$, q-Bessel Functions, especially Ramanujan (or q-Airy) function

$$\mathfrak{F}\left(\left\{z^{1/2}q^{(2k-1)/4}\right\}_{k=1}^{\infty}\right) = {}_0\phi_1\,(;0;q,-qz)$$

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- q-Confluent Hypergeometric Functions ${}_1\phi_1$

Proposition

Let $\{\lambda_n\}$ be positive and

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \left| \frac{w_n^2}{\lambda_n \lambda_{n+1}} \right| < \infty.$$

Then $A(z)$ is Hilbert-Schmidt for all $z \in \mathbb{C}$ and it holds

$$\det{}_2(1 + A(z)) = \mathfrak{F}\left(\left\{ \frac{\gamma_n^2}{\lambda_n - z} \right\}_{n=1}^{\infty} \right) \prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n}\right) e^{z/\lambda_n}$$

where the sequence $\{\gamma_n\}$ can be defined recursively as $\gamma_1 = 1$, $\gamma_{k+1} = w_k/\gamma_k$.

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where the sequence $\{\gamma_n\}$ can be defined recursively as $\gamma_1 = 1$, $\gamma_{k+1} = w_k/\gamma_k$.

- In the following we focus just on the function

$$F_J(z) := \mathfrak{F}\left(\left\{ \frac{\gamma_n^2}{\lambda_n - z} \right\}_{n=1}^{\infty} \right).$$

Characteristic function of complex Jacobi matrix

- Function F_J is well defined on $\mathbb{C} \setminus \overline{\{\lambda_n\}}$ if

$$\left\{ \frac{\gamma_n^2}{\lambda_n - z} \right\}_{n=1}^{\infty} \in D \quad \text{for all } z \in \mathbb{C} \setminus \overline{\{\lambda_n\}}$$

which holds if there is at least one $z_0 \in \mathbb{C} \setminus \overline{\{\lambda_n\}}$ such that

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(λ_n and w_n are complex!)

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- This assumptions is assumed everywhere from now.

- Function F_J is well defined on $\mathbb{C} \setminus \overline{\{\lambda_n\}}$ if
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which holds if there is at least one $z_0 \in \mathbb{C} \setminus \overline{\{\lambda_n\}}$ such that
$$\sum_{n=1}^{\infty} \left| \frac{w_n^2}{(\lambda_n - z_0)(\lambda_{n+1} - z_0)} \right| < \infty$$
(λ_n and w_n are complex!)
- This assumptions is assumed everywhere from now.
- F_J is meromorphic function on $\mathbb{C} \setminus \operatorname{der}(\{\lambda_n\})$ with poles in $z \in \{\lambda_n\} \setminus \operatorname{der}(\{\lambda_n\})$ of finite order less or equal to the number
$$r(z) := \sum_{n=1}^{\infty} \delta_{z,\lambda_n}.$$

Characteristic function of complex Jacobi matrix

- $$\mathfrak{Z}(J) := \left\{ z \in \mathbb{C} \setminus \mathrm{der}(\{\lambda_n\});\ \lim_{u \to z} (u - z)^{r(z)} F_J(u) = 0 \right\},$$

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- for $k \in \mathbb{Z}_+$ and $z \in \mathbb{C} \setminus \mathrm{der}(\{\lambda_n\})$, we put
$$\xi_k(z) := \lim_{u \to z} (u - z)^{r(z)} \left(\prod_{l=1}^{k} \frac{w_{l-1}}{u - \lambda_l} \right) \mathfrak{F}\left(\left\{ \frac{\gamma_l^2}{\lambda_l - u} \right\}_{l=k+1}^{\infty} \right)$$
where we set $w_0 := 1$.

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Theorem

Equalities

$$\mathrm{spec}(J) \setminus \mathrm{der}(\{\lambda_n\}) = \mathrm{spec}_p(J) \setminus \mathrm{der}(\{\lambda_n\}) = \mathfrak{Z}(J)$$

hold and, for $z \in \mathfrak{Z}(J)$,

$$\xi(z) := (\xi_1(z), \xi_2(z), \xi_3(z), \ldots)$$

is the eigenvector for eigenvalue z. Moreover, for $z \notin \overline{\{\lambda_n\}}$, vector $\xi(z)$ satisfies the formula

$$\sum_{k=1}^{\infty} (\xi_k(z))^2 = \xi_0'(z)\xi_1(z) - \xi_0(z)\xi_1'(z).$$

Green Function

- The Green function $G_{ij}(z) = (e_i, (J - z)^{-1} e_j)$ of J is expressible in terms of $\mathfrak{F}$,

$$G_{ij}(z) = -\frac{1}{w_M} \prod_{l=m}^{M} \left(\frac{w_l}{z - \lambda_l} \right) \frac{\mathfrak{F}\left(\left\{ \frac{\gamma_l^2}{\lambda_l - z} \right\}_{l=1}^{m-1} \right) \mathfrak{F}\left(\left\{ \frac{\gamma_l^2}{\lambda_l - z} \right\}_{l=M+1}^{\infty} \right)}{\mathfrak{F}\left(\left\{ \frac{\gamma_l^2}{\lambda_l - z} \right\}_{l=1}^{\infty} \right)}$$

where $z \in \rho(J)$, $m := \min(i, j)$, and $M := \max(i, j)$.

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  where $z \in \rho(J)$, $m := \min(i,j)$, and $M := \max(i,j)$.
- Especially, we get a compact formula for the Weyl m-function $m(z) = G_{11}(z)$,

$$m(z) = \frac{\mathfrak{F}\left(\left\{\frac{\gamma_j^2}{\lambda_j - z}\right\}_{j=2}^{\infty}\right)}{(\lambda_1 - z)\mathfrak{F}\left(\left\{\frac{\gamma_j^2}{\lambda_j - z}\right\}_{j=1}^{\infty}\right)}.$$

Examples with concrete operators – Bessel functions

- Let $\lambda_n = \alpha n$, $\alpha \neq 0$ and $w_n = w \neq 0$, $n = 1, 2, \ldots$. With this choice one has

$$J = \begin{pmatrix} \alpha & w & & & \\ w & 2\alpha & w & & \\ & w & 3\alpha & w & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

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- The characteristic function can be expressed as

$$F_J(z) = \left(\alpha^{-1} w\right)^{\alpha^{-1} z} \Gamma\left(1 - \alpha^{-1} z\right) J_{-\alpha^{-1} z}\left(2\alpha^{-1} w\right).$$

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$$v_k(z) = (-1)^k J_{k-\alpha^{-1}z}\left(2\alpha^{-1}w\right).$$

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- These results has been observed by many authors before [Gard & Zakrajšek, 1973].

- Let $q \in (0,1)$, $\beta \neq 0$, $\lambda_n = q^{n-1}$, and $w_n = \beta q^{(n-1)/2}$. Then matrix J has the form

$$J = \begin{pmatrix} 1 & \beta & & & \\ \beta & q & \beta\sqrt{q} & & \\ & \beta\sqrt{q} & q^2 & \beta q & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

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- The k-entry of the eigenvector corresponding to eigenvalue z^{-1} reads

$$v_k(z^{-1}) = q^{(k-1)(k-2)/4} \, (\beta z)^{k-1} \, (zq^k\,;q)_\infty \, {}_0\phi_1\Big(;zq^k;q,-q^k\beta^2 z^2\Big).$$

- In particular, for the characteristic function in the case $\lambda = 0$ and $w \in \ell^2(\mathbb{N})$, it holds

$$F_J(z^{-1}) = \mathfrak{F}\left(\left\{z\gamma_n^2\right\}_{n=1}^{\infty}\right) = \sum_{m=0}^{\infty}(-z^2)^m \sum_{k_1=1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \dots \sum_{k_m=k_{m-1}+2}^{\infty} w_{k_1}^2 w_{k_2}^2 \dots w_{k_m}^2 .$$

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Hence the spectrum of J is symmetric with respect to the origin.

- Let $w_n = 1/\sqrt{(n+\nu)(n+\nu+1)}$, $\nu \notin -\mathbb{N}$, then

$$F_J(z^{-1}) = \Gamma(\nu+1) z^{-\nu} J_\nu(2z).$$

and

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- Let $w_n = q^{n/2}$, $0 < q < 1$, then

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- For $x > 1$, $y \in \mathbb{R}$, put

$$\lambda(x,y) = \frac{y}{(x-1)x}, \quad w(x,y) = \frac{1}{x}\sqrt{\frac{x^2+y^2}{4x^2-1}}.$$

Examples with concrete operators – Confluent hypergeometric function

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- It holds

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- Thus we have

$$\mathrm{spec}(J(\mu,\nu)) = \left\{z^{-1};\ e^{-iz}\,{}_1F_1(\mu+i\nu;2\mu;2iz)=0\right\} \cup \{0\}$$

and

$$v_n(z^{-1}) = \sqrt{2\mu+2n-1}\,\frac{|\Gamma(\mu+n+i\nu)|}{\Gamma(2\mu+2n)}\,(2z)^{n-1}\,e^{-iz}\,{}_1F_1(\mu+n+i\nu;2\mu+2n;2iz).$$

Examples with concrete operators – Coulomb wave function

- In fact, one can show the characteristic function $F_J(z^{-1})$ is proportional to $F_{\mu-1}(-\nu, z^{-1})$ where function $F_L(\eta, \rho)$ is regular (at the origin) solution of second-order differential equation

$$\frac{d^2u}{d\rho^2} + \left[1 - \frac{2\eta}{\rho} - \frac{L(L+1)}{\rho^2}\right] u = 0,$$

  known as regular Coulomb wave function [Abramowitz&Stegun].
- Consequently, the spectrum of the corresponding Jacobi operator coincides with the set of reciprocal values of zeros of regular Coulomb wave function (as function of ρ).
- This has been originally observed by [Ikebe, 1975].

- For $\delta, a \in \mathbb{C}$, and $|q| < 1$, put

$$\lambda_n = (a+1)q^{n-1}, \quad w_n^2 = -aq^{n+\delta-1}(1-q^{n-\delta}).$$

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$$F_J(z^{-1}) = \frac{(z;q)_\infty}{((a+1)z;q)_\infty} {}_1\phi_1\left(zq^\delta, z; q, az\right).$$

Examples with concrete operators – q-Confluent hypergeometric function

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- Especially, for $\delta = 0$, the identity simplifies to

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- For $a > 0$, the operator J is not hermitian, however, $\mathrm{spec}(J)$ is real!

Theorem

In the ring of formal power series in the variables $t_1, \dots, t_n$, one has

$$\log \mathfrak{F}(t_1, \dots, t_n) = -\sum_{\ell=1}^{n-1} \sum_{m \in \mathbb{N}^\ell} \alpha(m) \sum_{k=1}^{n-\ell} \prod_{j=1}^{\ell} \left(t_{k+j-1} t_{k+j}\right)^{m_j}.$$

For a complex sequence $x = \{x_k\}_{k=1}^{\infty}$ such that $\sum_{k=1}^{\infty} |x_k x_{k+1}| < \log 2$ one has

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The proof is based on identity

$$\det \exp(A) = \exp(\operatorname{Tr} A), \quad A \in \mathbb{C}^{n,n}$$

together with formula relating $\mathfrak{F}(t_1, \dots, t_n)$ with determinant of a tridiagonal matrix depending on $t_1, \dots, t_n$.

The Hadamard factorization

- Let $\{x_n\}_{n=1}^{\infty} \subset \mathbb{C}$ such that $\sum_{n\geq 1} |x_n x_{n+1}| < \infty$ then we have

$$\mathfrak{F}\left(\{zx_n\}_{n=1}^{\infty}\right) = \det_2(1 - zJ)$$

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- Let $\{\xi_n\}_{n=1}^{\Omega}$ denotes zeros of the even function
$$z \mapsto \mathfrak{F}\left(\{zx_n\}_{n=1}^{\infty}\right)$$
with non-negative real parts. Moreover, we arrange these zeros in the ascending order of their modulus.

The Hadamard factorization

- Let $\{x_n\}_{n=1}^{\infty} \subset \mathbb{C}$ such that $\sum_{n\geq 1} |x_n x_{n+1}| < \infty$ then we have

$$\mathfrak{F}\left(\{zx_n\}_{n=1}^{\infty}\right) = \det_2(1 - zJ)$$

  where J is Jacobi operator with vanishing diagonal and off-diagonal $w_n = \sqrt{x_n x_{n+1}}$.
- Since we assume $w \in \ell^2(\mathbb{N})$ operator J is Hilbert-Schmidt and as it is well known the regularized determinant admits the Hadamard factorization into the (infinite) product.
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Theorem

Let $\sum_{n\geq 1} |x_n x_{n+1}| < \infty$ then it holds

$$\mathfrak{F}\left(\{zx_n\}_{n=1}^{\infty}\right) = \prod_{n=1}^{\Omega}\left(1 - \frac{z^2}{\xi_n^2}\right).$$

Main topic

Notation:

- For a multiindex $m \in \mathbb{N}^{\ell}$ denote by $|m| = \sum_{j=1}^{\ell} m_j$ its order and by $d(m) = \ell$ its length.

Formula for the spectral zeta function

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- For $N \in \mathbb{N}$ define

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Corollary

For any $n \in \mathbb{N}$,

$$\zeta_J(2n) = \sum_{k=1}^{\Omega} \frac{1}{\xi_k^{2n}} = n \sum_{m \in \mathcal{M}(n)} \alpha(m) \sum_{k=1}^{\infty} \prod_{j=1}^{d(m)} \left(x_{k+j-1} x_{k+j}\right)^{m_j}.$$

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To prove the identity one has to apply logarithm on both sides of
$$\mathfrak{F}\left(\{z x_n\}_{n=1}^{\infty} \right) = \prod_{n=1}^{\Omega} \left(1 - \frac{z^2}{\xi_n^2} \right),$$
use the formula for the logarithm of $\mathfrak{F}$ and equate coefficients at the same power of z.

- By using the spectral zeta function one can localize the largest eigenvalue in modulus (the spectral radius) of Hermitian *J* since

$$\frac{\zeta_J(2n+2)}{\zeta_J(2n)} < \frac{1}{\xi_1^2} < \sqrt[n]{\zeta_J(2n)}.$$

Application of the spectral zeta function

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- In fact, the inequalities become equalities in the limit $n \to \infty$. Thus, one can even obtain an explicit formula

$$\frac{1}{\xi_1} = \lim_{N\to\infty} \left(\sum_{m\in\mathcal{M}(N)} \alpha(m) \sum_{k=1}^{\infty} \prod_{j=1}^{d(m)} \left(x_{k+j-1}x_{k+j}\right)^{m_j} \right)^{1/(2N)} .$$

Two examples – Rayleigh special function

- Put $x_n = (2\nu + 2n)^{-1}$, where $\nu > -1$. Then, as a particular case of the factorization theorem, one has

$$\mathfrak{F}\left(\{zx_n\}_{n=1}^{\infty}\right) = \Gamma(\nu+1)\left(\frac{z}{2}\right)^{-\nu} J_\nu(z) = \prod_{k=1}^{\infty}\left(1 - \frac{z^2}{j_{\nu,k}^2}\right)$$

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- The zeta function

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- Its values $\sigma_\nu(2N)$ for $N \in \mathbb{N}$ are rational functions in ν.
  – originally computed by Rayleigh for $1 \leq N \leq 5$;
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- Its values $\sigma_\nu(2N)$ for $N \in \mathbb{N}$ are rational functions in ν.
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- The general formula reads

$$\sigma_\nu(2N) = 2^{-2N} N \sum_{k=1}^{\infty} \sum_{m \in \mathcal{M}(N)} \alpha(m) \prod_{j=1}^{d(m)} \left(\frac{1}{(j+k+\nu-1)(j+k+\nu)}\right)^{m_j}.$$

Two examples – Zeta function associated with q-Airy function

- Put $x_n = q^{(2n-1)/4}$, where $0 < q < 1$. Then we have

$$\mathfrak{F}\left(\left\{wq^{(2n-1)/4}\right\}_{n=1}^{\infty}\right) = A_q(w^2)$$

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where A_q is the q-Airy function.
- Zeros of $A_q(z)$ are exactly $0 < \iota_1(q) < \iota_2(q) < \iota_3(q) < \ldots$, all of them are simple and
$$A_q(z) = \prod_{k=1}^{\infty}\left(1 - \frac{z}{\iota_k(q)}\right).$$

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$$A_q(z) = \prod_{k=1}^{\infty}\left(1 - \frac{z}{\iota_k(q)}\right).$$

- Formula for integer values of the Rayleigh-like function associated with $A_q(z)$, denoted as $Z_n(q)$, reads

$$Z_n(q) := \sum_{k=1}^{\infty}\frac{1}{\iota_k(q)^n} = \frac{nq^n}{1-q^n}\sum_{m\in\mathcal{M}(n)}\alpha(m)\,q^{\epsilon_1(m)},$$

  where

$$\forall m \in \mathbb{N}^{\ell},\ \epsilon_1(m) = \sum_{j=1}^{\ell}(j-1)\,m_j.$$

1. F. Štampach, P. Šťovíček: *The characteristic function for Jacobi matrices with applications*, Linear Alg. Appl. **438** (2013) 4130-4155.
2. F. Štampach, P. Šťovíček: *Special functions and spectrum of Jacobi matrices*, Linear Algebra Appl. (2013) (in press).
3. F. Štampach, P. Šťovíček: *A logarithm formula and factorization of the characteristic function of a Jacobi matrix* , (2013) (preprint).

Preprints are available on arXiv or at websites `http://users.fit.cvut.cz/~stampfra/`.

Thank you!