One-parameter generalization of Charlier and Al-Salam-Carlitz polynomials

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Generalized Charlier polynomials

For $n=-1,0,1,\dots$, we define polynomials $P_n(\alpha,\beta;x)$ recursively as the solution of recurrence

$$u_{n+1}=(x-n-\beta)u_n-(\alpha+\beta n)u_{n-1}, \tag{1}$$

with initial setting $P_{-1}(\alpha,\beta;x)=0$ and $P_0(\alpha,\beta;x)=1$, α and β being complex parameters.
For the second linearly independent solution $Q_n(\alpha,\beta;x)$ of (1) it holds

$$Q_n(\alpha,\beta;x)=P_{n-1}(\alpha+\beta,\beta;x-1),\quad n\in\mathbb{Z}_+.$$

Polynomials $P_n(\alpha,\beta;x)$ can be viewed as a one-parameter generalization of Charlier orthogonal polynomials (see Askey scheme). Namely, for $\alpha=0$, polynomials $P_n(0,\beta;x)$ coincide with Charlier polynomials $C_n^{(\beta)}(x)$, which can be expressed in closed explicit form as polynomials in β,

$$C_n^{(\beta)}(x)=\sum_{k=0}^{n}\binom{n}{k}\binom{x}{k}k!(-\beta)^{n-k},\quad n\in\mathbb{Z}_+.$$

On the other hand, by setting $\beta=0$, one obtains a relation between $P_n(\alpha,0;x)$ and Lommel polynomials, namely,

$$P_n(\alpha,0;x)=(-1)^n\alpha^{\frac{n}{2}}R_{n,-x}(2\sqrt{\alpha}),\quad n\in\mathbb{Z}_+.$$

Hypergeometric functions and asymptotics

For $n=-1,0,1,\dots$, we have an expression of polynomials $P_n(\alpha,\beta;x)$ in terms of confluent hypergeometric functions,

$$\begin{aligned}P_n(\alpha,\beta;x)=e^{-\beta}\Bigg[&\frac{\Gamma(x+1)}{\Gamma(x+1-n)}\,{}_1F_1\left(-\frac{\alpha}{\beta}-x;-x;\beta\right){}_1F_1\left(-\frac{\alpha}{\beta}-n;x-n+1;\beta\right)\\ &-\beta^{n+1}\frac{\Gamma\left(n+1+\frac{\alpha}{\beta}\right)\Gamma(x-n)}{\Gamma\left(\frac{\alpha}{\beta}\right)\Gamma(x+2)}\,{}_1F_1\left(1-\frac{\alpha}{\beta};x+2;\beta\right){}_1F_1\left(-\frac{\alpha}{\beta}-x;-x+n+1;\beta\right)\Bigg].\end{aligned}$$

From the previous formula one deduces the leading term of the asymptotic expansion of $P_n(\alpha,\beta;x)$, for $n\to\infty$, getting

$$P_n(\alpha,\beta;x)=(-1)^n\frac{\Gamma(n-x)}{\Gamma(-x)}\,{}_1F_1\left(-\frac{\alpha}{\beta}-x;-x;\beta\right)(1+o(1)).$$

Orthogonality relation

Let $\beta>0$ and $\alpha+\beta>0$. Then it holds:

i) Polynomials $\{P_n(\alpha,\beta;x):n\in\mathbb{Z}_+\}$ satisfy the orthogonality relation

$$\int_{\mathbb{R}}P_n(\alpha,\beta;x)P_m(\alpha,\beta;x)d\mu(x)=\beta^n\frac{\Gamma\left(\frac{\alpha}{\beta}+n+1\right)}{\Gamma\left(\frac{\alpha}{\beta}+1\right)}\delta_{mn},\quad m,n\in\mathbb{Z}_+,$$

where $d\mu$ is a purely discrete probability measure.

ii) $d\mu$ is supported by the set

$$\operatorname{supp}(d\mu)=\left\{x\in\mathbb{R}:\frac{1}{\Gamma(-x)}\,{}_1F_1\left(-\frac{\alpha}{\beta}-x;-x;\beta\right)=0\right\},$$

and the step function $\mu(x)$ has jumps at $x\in\operatorname{supp}(d\mu)$ of magnitude

$$\mu(x)-\mu(x-0)=\frac{{}_1F_1\left(-\frac{\alpha}{\beta}-x;1-x;\beta\right)}{x\frac{\partial}{\partial x}\,{}_1F_1\left(-\frac{\alpha}{\beta}-x;-x;\beta\right)}$$

(assuming distribution function $\mu(x)$ is continuous from the right).

iii) The Stieltjes transform of $d\mu$ is given by

$$\int_{\mathbb{R}}\frac{d\mu(x)}{z-x}=-\frac{{}_1F_1\left(-\frac{\alpha}{\beta}-z;1-z;\beta\right)}{z\,{}_1F_1\left(-\frac{\alpha}{\beta}-z;-z;\beta\right)},$$

for $z\notin\operatorname{supp}(d\mu)$.

Generating function and structure relation

If $\alpha/\beta>0$ then the generating function of polynomials $P_n(\alpha,\beta;x)$ has the integral representation,

$$\sum_{n=0}^{\infty}\frac{P_n(\alpha,\beta;x)}{\Gamma(n+1+\alpha/\beta)}w^n=\frac{e^{-\beta w}w^{-\alpha/\beta}(1+w)^{x+\alpha/\beta}}{\Gamma(\alpha/\beta)}\int_0^w e^{\beta t}t^{-1+\alpha/\beta}(1+t)^{-1-x-\alpha/\beta}dt,$$

for $|w|<1$. Further, a kind of forward shift formula for $P_n(\alpha,\beta;x)$ reads

$$P_n(\alpha,\beta;x+1)-P_n(\alpha,\beta;x)=\left(n+\frac{\alpha}{\beta}\right)P_{n-1}(\alpha,\beta;x)-\frac{\alpha}{\beta}P_{n-1}(\alpha+\beta,\beta;x).$$

Generalized Al-Salam-Carlitz I polynomials

For $n=-1,0,1,\dots$, we define polynomials $U_n(a,\delta;q,x)$ recursively as the solution of recurrence

$$v_{n+1}=\left(x-(a+1)q^n\right)v_n+aq^{n+\delta-1}(1-q^{n-\delta})v_{n-1}, \tag{2}$$

with initial setting $U_{-1}(a,\delta;q,x)=0$ and $U_0(a,\delta;q,x)=1$. Parameters a and δ are assumed to be complex, in general, and $0<q<1$.
For the second linearly independent solution $V_n(a,\delta;q,x)$ of (2) it holds

$$V_n(a,\delta;q,x)=q^{n-1}U_{n-1}(a,\delta-1;q,q^{-1}x),\quad n\in\mathbb{Z}_+.$$

Polynomials $U_n(a,\delta;q,x)$ can be viewed as a one-parameter generalization of Al-Salam-Carlitz I orthogonal polynomials (see q-Askey scheme). Namely, for $\delta=0$, polynomials $U_n(a,0;q,x)$ coincide with Al-Salam-Carlitz I polynomials $U_n^{(a)}(x;q)$, which can be expressed in terms of q-hypergeometric function as

$$U_n^{(a)}(x;q)=(-a)^nq^{\binom{n}{2}}\,{}_2\phi_1\left(q^{-n},x^{-1};0;q,a^{-1}qx\right),\quad n\in\mathbb{Z}_+.$$

q-hypergeometric functions and asymptotics

For $n=-1,0,1,\dots$ and $x\notin\{0\}\cup q^{\mathbb{Z}}\cup aq^{\delta+\mathbb{Z}_+}$, we have an expression of polynomials $U_n(a,\delta;q,x)$ in terms of q-confluent hypergeometric functions,

$$\begin{aligned}U_n(a,\delta;q,x)=\frac{1}{(ax^{-1}q^\delta;q)_\infty}\Bigg[&x^n(x^{-1};q)_n\,{}_1\phi_1\left(x^{-1}q^\delta;x^{-1};q,ax^{-1}\right){}_1\phi_1\left(q^{\delta-n};q^{1-n}x;q,aq\right)\\ &-\frac{(-a)^{n+1}q^{\delta(n+1)-1+n(n-1)/2}(q^{-\delta};q)_{n+1}}{x^{n+2}(x^{-1}q^{-1};q)_{n+2}}\,{}_1\phi_1\left(x^{-1}q^\delta;x^{-1}q^{n+1};q,ax^{-1}q^{n+1}\right)\\ &\times{}_1\phi_1\left(q^{\delta+1};q^2x;q,aq\right)\Bigg].\end{aligned}$$

From the previous formula one deduces the leading term of the asymptotic expansion of $U_n(a,\delta;q,x)$, whenever $x\neq0$, for $n\to\infty$, getting

$$U_n(a,\delta;q,x)=x^n(x^{-1};q)_n\,{}_1\phi_1\left(x^{-1}q^\delta;x^{-1};q,ax^{-1}\right)(1+o(1)).$$

Orthogonality relation

Let $a,\delta<0$ and $0<q<1$. Then it holds:

i) Polynomials $\{U_n(a,\delta;q,x):n\in\mathbb{Z}_+\}$ satisfy the orthogonality relation

$$\int_{\mathbb{R}}U_m(a,\delta;q,x)U_n(a,\delta;q,x)d\mu(x)=(-a)^nq^{n\delta+n(n-1)/2}(q^{-\delta};q)_{n+1}\delta_{mn}$$

where $d\mu$ is a discrete positive measure.

ii) $d\mu$ is supported by the set

$$\operatorname{supp}(d\mu)=\left\{x^{-1}\in\mathbb{R}:\,{}_2\phi_1\left(axq^\delta,xq^\delta;0;q,q^{-\delta}\right)=0\right\}\cup\{0\},$$

and the step function $\mu(x)$ has jumps at $x\in\operatorname{supp}(d\mu)\setminus\{0\}$ of magnitude

$$\mu(x)-\mu(x-0)=\frac{{}_2\phi_1\left(ax^{-1}q^\delta,x^{-1}q^\delta;0;q,q^{1-\delta}\right)}{x\frac{\partial}{\partial x}\left[{}_2\phi_1(ax^{-1}q^\delta,x^{-1}q^\delta;0;q,q^{-\delta})\right]}$$

(assuming distribution function $\mu(x)$ is continuous from the right).

iii) The Stieltjes transform of $d\mu$ is given by

$$\int_{\mathbb{R}}\frac{d\mu(x)}{z-x}=\frac{{}_2\phi_1\left(az^{-1}q^\delta,z^{-1}q^\delta;0;q,q^{1-\delta}\right)}{z\,{}_2\phi_1(az^{-1}q^\delta,z^{-1}q^\delta;0;q,q^{-\delta})},$$

for $z\notin\operatorname{supp}(d\mu)$.

Generating function and structure relation

Let $a\in\mathbb{R}$ and $x\neq0$ then the generating function for $U_n(a,\delta;q,x)$ reads:

i) if $\delta<0$,

$$\sum_{n=0}^{\infty}\frac{U_n(a,\delta;q,x)}{(q^{1-\delta};q)_n}t^n=(1-q^{-\delta})\sum_{k=0}^{\infty}\frac{(aq^\delta t;q)_k(q^\delta t;q)_k}{(xt;q)_{k+1}}q^{-k\delta},\quad |xt|<1,$$

ii) if $\delta=0$ (Al-Salam-Carlitz case),

$$\sum_{n=0}^{\infty}\frac{U_n(a,0;q,x)}{(q;q)_n}t^n=\frac{(at;q)_\infty(t;q)_\infty}{(xt;q)_\infty},\quad |xt|<1.$$

Further, a kind of forward shift formula for $U_n(a,\delta;q,x)$ reads

$$\begin{aligned}&U_n(a,\delta;q,x)-U_n(a,\delta;q,qx)\\ &\quad=x(1-q^{n-\delta})U_{n-1}(a,\delta;q,x)-xq^n(1-q^{-\delta})U_{n-1}(a,\delta-1;q,x).\end{aligned}$$