On a Characteristic Function of a Certain Class of Jacobi Matrices

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Functions $\mathfrak{E}$, $\mathfrak{F}$

Definition

Define $\mathfrak{E} : D \to \mathbb{C}$ and $\mathfrak{F} : D \to \mathbb{C}$

$$\mathfrak{E}(x) = 1 + \sum_{m=1}^{\infty} \sum_{k_1=1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \ldots \sum_{k_m=k_{m-1}+2}^{\infty} x_{k_1} x_{k_1+1} x_{k_2} x_{k_2+1} \ldots x_{k_m} x_{k_m+1},$$

$$\mathfrak{F}(x) = 1 + \sum_{m=1}^{\infty} (-1)^m \sum_{k_1=1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \ldots \sum_{k_m=k_{m-1}+2}^{\infty} x_{k_1} x_{k_1+1} x_{k_2} x_{k_2+1} \ldots x_{k_m} x_{k_m+1},$$

where

$$D = \left\{ \{x_k\}_{k=1}^{\infty} \subset \mathbb{C};\ \sum_{k=1}^{\infty} |x_k x_{k+1}| < \infty \right\}.$$

These functions obey a tree-term recurrence rule:

$$\mathfrak{F}(x) = \mathfrak{F}(x_1, \ldots, x_k)\, \mathfrak{F}(T^k x) - \mathfrak{F}(x_1, \ldots, x_{k-1}) x_k x_{k+1} \mathfrak{F}(T^{k+1} x),$$
$$\mathfrak{E}(x) = \mathfrak{E}(x_1, \ldots, x_k)\, \mathfrak{E}(T^k x) + \mathfrak{E}(x_1, \ldots, x_{k-1}) x_k x_{k+1} \mathfrak{E}(T^{k+1} x).$$

Relation between $\mathfrak{F}$ and the spectrum of a Jacobi matrix

- Denote J Jacobi matrix of the form

$$J := \begin{pmatrix} \lambda_1 & w_1 & & & \\ w_1 & \lambda_2 & w_2 & & \\ & w_2 & \lambda_3 & w_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

where $\{\lambda_n\}$ is real and $\{w_n\}$ is positive sequence. Let us assume

$$\sum_{n=1}^{\infty} \frac{w_n^2}{\lambda_n \lambda_{n+1}} < \infty \quad \text{a} \quad \lim_{n\to\infty} \lambda_n = +\infty.$$

Then the spectrum of J is discrete and simple.

- Define seq. $\{\gamma_n\}$ recursively as $\gamma_1 = 1$ and $\gamma_{k+1} = w_k/\gamma_k$.
- We focus on function

$$\mathfrak{F}\left(\left\{\frac{\gamma_k^2}{\lambda_k - z}\right\}_{k=1}^{\infty}\right)$$

which is analytic on $\mathbb{C} \setminus \{\lambda_k\}$ and it has poles in $z = \lambda_k$ of finite order $r_k = \sum_{n\geq 1} \delta_{(\lambda_n,\lambda_k)}$.

Zeros of $\mathfrak{F}$ as eigenvalues of J

Theorem

Let J is self-adjoint and the conditions

$$\sum_{n=1}^{\infty} \frac{w_n^2}{\lambda_n \lambda_{n+1}} < \infty \quad \text{a} \quad \lim_{n\to\infty} \lambda_n = +\infty$$

are satisfied. Next, let

$$\lim_{n\to\infty} \frac{w_n w_{n-1}}{\lambda_n} = 0.$$

Then

$$z \in \operatorname{spec}(J) \setminus \{\lambda_n\}_{n=1}^{\infty} \iff \mathfrak{F}\left(\left\{\frac{\gamma_k^2}{\lambda_k - z}\right\}_{k=1}^{\infty}\right) = 0$$

and

$$\lambda_s \in \operatorname{spec}(J) \iff \lim_{z\to\lambda_s} (\lambda_s - z)^{r_s} \mathfrak{F}\left(\left\{\frac{\gamma_k^2}{\lambda_k - z}\right\}_{k=1}^{\infty}\right) = 0,$$

where $r_s = \sum_{n\geq 1} \delta_{(\lambda_n, \lambda_s)}$.