Some Results on the Spectrum of Complex Jacobi Operators

F. Štampach†✉, and P. Šťovíček†

†Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University in Prague

✉stampfra@fjfi.cvut.cz

Introduction

For $\lambda \equiv \{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{C}$ and $w \equiv \{w_n\}_{n=1}^{\infty} \subset \mathbb{C} \setminus \{0\}$, let us denote

$$\mathcal{J} := \begin{pmatrix} \lambda_1 & w_1 & & & \\ w_1 & \lambda_2 & w_2 & & \\ & w_2 & \lambda_3 & w_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Further, we denote by J an operator acting on $\ell^2(\mathbb{N})$ by formal matrix product $Jx := \mathcal{J}x$ with domain $\mathrm{Dom}(J) = \{x \in \ell^2(\mathbb{N}) : \mathcal{J}x \in \ell^2(\mathbb{N})\}$. Next, we restrict ourself only on a particular class of Jacobi operators satisfying the following conditions:

(A1) the resolvent set $\rho(J)$ of J is nonempty (i.e., $\mathcal{J}$ is proper, see [Be]);

(A2) there exists at least one $z \in \mathbb{C} \setminus \mathrm{der}(\lambda)$ such that

$$\sum_{n=1}^{\infty} \left| \frac{w_n^2}{(\lambda_n - z)(\lambda_{n+1} - z)} \right| < \infty.$$

($\mathrm{der}(\lambda)$ stands for the set of all accumulation points of the sequence λ.)

Function $\mathfrak{F}$

Definition 1 *Let us define $\mathfrak{F} : D \to \mathbb{C}$ by the relation*

$$\mathfrak{F}(x) = 1 + \sum_{m=1}^{\infty} (-1)^m \sum_{k_1=1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \dots \sum_{k_m=k_{m-1}+2}^{\infty} x_{k_1} x_{k_1+1} x_{k_2} x_{k_2+1} \dots x_{k_m} x_{k_m+1}$$

where

$$D = \left\{ \{x_k\}_{k=1}^{\infty} \subset \mathbb{C};\ \sum_{k=1}^{\infty} |x_k x_{k+1}| < \infty \right\}.$$

For a finite number of complex variables we identify $\mathfrak{F}(x_1, x_2, \dots, x_n)$ with $\mathfrak{F}(x)$ where $x = (x_1, x_2, \dots, x_n, 0, 0, 0, \dots)$. By convention, we also put $\mathfrak{F}(\emptyset) = 1$ where $\emptyset$ is the empty sequence.

Note the domain D is not a linear space. One has, however, $\ell^2(\mathbb{N}) \subset D$.

Properties of $\mathfrak{F}$

The function $\mathfrak{F}$ restricted on $\ell^2(\mathbb{N})$ is a continuous functional on $\ell^2(\mathbb{N})$. Moreover, for $k = 1, 2, \dots$, this function obey a three-term recurrence relation

$$\mathfrak{F}(x) = \mathfrak{F}(x_1, \dots, x_k)\, \mathfrak{F}(T^k x) - \mathfrak{F}(x_1, \dots, x_{k-1}) x_k x_{k+1} \mathfrak{F}(T^{k+1} x)$$

and, furthermore, it holds

$$\lim_{n\to\infty} \mathfrak{F}(T^n x) = 1, \qquad \lim_{n\to\infty} \mathfrak{F}(x_1, x_2, \dots, x_n) = \mathfrak{F}(x),$$

for any $x \in D$. T stands for the shift operator, $T(x_1, x_2, \dots) = (x_2, x_3, \dots)$.

Some special functions are expressible in terms of $\mathfrak{F}$, first of all the Bessel function of the first kind. More precisely, for $\nu \notin -\mathbb{N}$, one has

$$J_\nu(2w) = \frac{w^\nu}{\Gamma(\nu+1)} \mathfrak{F}\left(\left\{ \frac{w}{\nu + k} \right\}_{k=1}^{\infty} \right)$$

where J_ν stands for the Bessel function of the first kind.

Next, let us define a sequence $\{\gamma_k\}_{k=1}^{\infty}$ recursively by

$$\gamma_1 = 1, \quad \gamma_{k+1} = w_k / \gamma_k \quad \text{for} \quad k = 1, 2, \dots.$$

Finally, the characteristic function of a finite Jacobi matrix J_N which is $N \times N$ truncation of $\mathcal{J}$ can be expressed with the aid of $\mathfrak{F}$.

Theorem 2 *Let $\{\gamma_k\}_{k=1}^{N}$ be the sequence defined above and $z \in \mathbb{C}$. Then it holds*

$$\det(J_N - zI_N) = \left(\prod_{k=1}^{N} (\lambda_k - z) \right) \mathfrak{F}\left(\frac{\gamma_1^2}{\lambda_1 - z}, \frac{\gamma_2^2}{\lambda_2 - z}, \dots, \frac{\gamma_N^2}{\lambda_N - z} \right).$$

A Characteristic Function

Next, we focus on the function

$$F_J(z) := \mathfrak{F}\left(\left\{ \frac{\gamma_k^2}{\lambda_k - z} \right\}_{k=1}^{\infty} \right)$$

considered as a complex function of one complex variable z. This function is well defined (due to assumption (A2)) and analytic on $\mathbb{C} \setminus \overline{\lambda}$ ($\overline{\lambda}$ is the closure of λ), and it has poles in points $z \in \lambda \setminus \mathrm{der}(\lambda)$ of finite order less or equal to

$$r_z = \sum_{k=1}^{\infty} \delta_{(\lambda_k, z)}.$$

Further, we slightly extend definition of the function F_J. For $\xi \in \mathbb{C} \setminus \mathrm{der}(\lambda)$ and $m = 0, 1, 2, \dots$, let us define

$$F_{J,m}^{\xi}(z) := \begin{cases} (z-\xi)^{r_\xi} \mathfrak{F}\left(\left\{ \frac{\gamma_k^2}{\lambda_k - z} \right\}_{k=m+1}^{\infty} \right), & \text{if } z \neq \xi \\ \lim_{z\to\xi} (z-\xi)^{r_\xi} \mathfrak{F}\left(\left\{ \frac{\gamma_k^2}{\lambda_k - z} \right\}_{k=m+1}^{\infty} \right), & \text{if } z = \xi \end{cases}$$

If $m = 0$ we write $F_J^{\xi}(z)$ instead of $F_{J,0}^{\xi}(z)$. We call the function $F_J^{z}(z)$ a *characteristic function* of J.

Main results

The relation between the spectrum of J and the characteristic function are described in following theorems. For these purposes, let us denote

$$\xi_k(z) := \prod_{l=1}^{k} \left(\frac{w_{l-1}}{z - \lambda_l} \right) F_{J,k}^{z}(z) \qquad k = 0, 1, 2, \dots, \qquad (w_0 := 1).$$

Theorem 3 *Let condition (A2) is fulfilled and let $\mathrm{der}(\lambda)$ be bounded. If $\xi_0(z) \equiv F_J^z(z) = 0$ for some $z \in \mathbb{C} \setminus \mathrm{der}(\lambda)$, then z is an eigenvalue of J and vector $\xi(z) \equiv \{\xi_k(z)\}_{k=1}^{\infty}$ is the respective eigenvector.*

The following theorem describes a reverse statement.

Theorem 4 *Let conditions (A1), (A2) hold and let $\mathrm{der}(\lambda)$ be bounded. Further, let $z \notin \mathrm{der}(\lambda)$ is an isolated point of the spectrum of J. If z is an eigenvalue of J then $F_J^z(z) = 0$.*

Example By using Theorems 3 and 4, for J with $\lambda_n = n$, and $w_n = iw$, $w > 0$, one can find out

$$\mathrm{spec}(J) = \{ z \in \mathbb{C} : I_{-z}(2w) = 0 \}$$

where I is the modified Bessel function of the first kind.

Green function

Matrix elements of the resolvent $(J - z)^{-1}$ can be expressed in terms of $\mathfrak{F}$.

Theorem 5 *Let conditions (A1) and (A2) hold. Then, for $z \in \rho(J)$ and $k, l \in \mathbb{N}$, $k \leq l$, one has*

$$(e_l, (J-z)^{-1} e_k) = -\frac{1}{w_l} \prod_{j=k}^{l} \left(\frac{w_j}{z - \lambda_j} \right) \frac{\mathfrak{F}\left(\left\{ \frac{\gamma_j^2}{\lambda_j - z} \right\}_{j=1}^{k-1} \right) \mathfrak{F}\left(\left\{ \frac{\gamma_j^2}{\lambda_j - z} \right\}_{j=l+1}^{\infty} \right)}{\mathfrak{F}\left(\left\{ \frac{\gamma_j^2}{\lambda_j - z} \right\}_{j=1}^{\infty} \right)}$$

where $\{e_k\}_{k=1}^{\infty}$ is the canonical basis of $\ell^2(\mathbb{N})$.

Especially, one can use Theorem 5 to find a simple formula for the *Weyl m-function* $(e_1, (J-z)^{-1} e_1)$. This function turns out to be very useful for an investigation of isolated points of the spectrum of J.

Fundamental references

[Be] B. Beckerman: *Complex Jacobi matrices*, J. Comput. Appl. Math.,127, (2001), 17-65.

[SS] F. Štampach, P. Šťovíček: *On the eigenvalue problem for a particular class of finite Jacobi matrices*, Lin. Alg. App., 434, (2011), 1336-1353.