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presented:

1.5-approximation algorithm for the 2-Convex Recoloring problem

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Definitions

- Let $G = (V, E)$ be a graph and let $\chi : V \to C$ be a coloring function, assigning each vertex in V a color in C. We say that χ is a **convex coloring** of G if, for every color $c \in C$, the vertices with color c induce a connected sub-graph of G.
- Given a colored graph G_χ, if two vertices in the colored graph have the same color, we call them a **pair**, if they are connected with and edge, then they are a **connected pair**, otherwise they are a **disconnected pair**. Any vertex with a unique color c is a **singleton**. We denote by $G_\chi[c]$ the subgraph induced by the set of vertices $\{v : \chi(v) = c\}$
- Given a colored graph G_χ and recoloring χ', a vertex v **retains** its color if $\chi(v) = \chi'(v)$. We say that χ' retains a pair p, if both vertices of p retain their color. The recoloring χ' retains a color $c \in C$, if there exists a vertex $v \in G$ such that $\chi(v) = \chi'(v) = c$. If a recoloring retains all colors of a graph, we refer to it as a **retains-all recoloring**.
- Given a colored graph G_χ and convex recoloring χ', we say that χ' **path-recolors** G with respect to $c \in C$ if there is a Hamiltonian path in $G_{\chi'}[c'] : u, ..., v$ such that $\chi(u) = \chi(v) = c$. A special case of this definition is when $G_{\chi'}[c']$ is a single vertex v and $\chi(v) = c$. We say that χ' is a **path-recoloring** if χ' does not recolor any connected pair and χ' path-recolors G with respect to every $c \in C$.
- When recoloring χ' colors a path between a disconnected pair, we refer to the path as a colored path. Let D be the set of all disconnected pairs in G, and denote by I the set of colored path in $G_{\chi'}$.
- Given a colored gaph G_χ and path p let $V(p)$ be the set of vertices on the path and let $\chi(p)$ be the set of colors assigned to vertices on this path. Given two paths p_1 and p_2 in G_χ : p_1 and p_2 are in **direct conflict** if $V(p_1) \cap V(p_2) \neq \emptyset$. p_1 and p_2 are in **indirect conflict** if $\chi(p_1) \cap \chi(p_2) \neq \emptyset$. p_1 and p_2 are in **conflict** if they are in a direct or an indirect conflict. If two path are not in conflict, then they are **independent**. Given a set of path I, we say that this set is independent if it is pairwise independent. A path $v, ..., u$ in G is called **colorable** if u and v form a disconnected pair and the path does not conatin singletons nor vertices of connected pairs.
- l_p - number of vertices on a path p
- For a path $p \in I$, the set of path in I^* such that p is their conflict source is donted as $N(p)$. The members of $N(p)$ are called the neighbors of p. Denote $d_p := |N(p)|$ and refer to d_p as the degree of p

Problems

Convex Recoloring problem (CR)
Input: Colored graph G_χ
Output: Recoloring of a minimum number of vertices of G, such that the resulting coloring is convex.

t-Convex Recoloring problem (t-CR)
A special case, in wich the given coloring assigns the same color to at most t vertices in G.

Theorems

Theorem 1. The weighted version of 2-CR cannot be approximated within any multiplicative ration, unless $P = NP$

Lemma 1. For every colored graph G_χ, there exists a retains-all optimal convex recoloring.

Lemma 2. For every colored graph G_χ there exists a retains-all, optimal, convex recoloring that does not recolor any connected pair.

Lemma 3. For every colored graph G_χ there exists an optimal recoloring that is a path recoloring.

Lemma 4. Given a colored graph G_χ, a path-recoloring χ' recolors exactly $|D| - |I|$ vertices.

Lemma 5. Let G_χ be a colored graph. Also, let χ' be a path-recoloring, and let I denote the set of colored path. Then I is an independent set of colorable path with respect to χ.

Lemma 6. For any independent set of colorable path I in G_χ, there exists a path-recoloring χ' of G, where I is the set of colored paths.

Theorem 2. Given a colored graph G_χ, the cost of an optimal recoloring is $|D| - s$ if and only if the size of maximum independent set of colorable paths is s.
Lemma 7. If $p' \in I^* \setminus I$, then there is a path $p \in I$ that is in conflict with p' and $l_p \leq l_{p'}$.
Observation 1. For every path $p \in I$, if $p' \in N(p)$ then $l_p \leq l_{p'}$.
Lemma 8. For every path $p \in I$, $d_p \leq l_p - 1$
Lemma 9. $|D| \geq 2|I^*|$.
Observation 2. $\sum_{p \in I} d_p = |I^*| = \alpha.|I|$
Lemma 10. $\frac{\sum_{p \in I} d_p^2}{|I|} \geq \alpha^2$
Lemma 11. $|D| \geq \alpha^2 |I|$.
Theorem 3. The greedy algorithm is a $\frac{3}{2}$-approximation algorithm for 2-CR.