Spectral analysis of two doubly infinite Jacobi operators

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Spectral Theory and Applications

conference in memory of Boris Pavlov

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Contents

Two doubly-infinite Jacobi matrices

We analyze the spectral properties of Jacobi operators A and B acting on vectors of the standard basis of $\ell^2(\mathbb{Z})$ as:

$$Ae_n = q^{-n+1}e_{n-1} + q^{-n}e_{n+1}, \quad n \in \mathbb{Z},$$

and

$$Be_n = e_{n-1} + \alpha q^{-n}e_n + e_{n+1}, \quad n \in \mathbb{Z},$$

where $q \in (0, 1)$ and $\alpha \in \mathbb{R}$.

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*The spectrum of any associated semi-infinite Jacobi operator is never known explicitly ($\alpha \neq 0$) but is expressible in terms of zeros of certain special functions.*functions

- Let $0 < q < 1$, $r, s \in \mathbb{Z}_+$. Recall the basic hypergeometric function

$$ {}_r\phi_s \begin{bmatrix} a_1, & a_2, & \dots & a_r \\ b_1, & b_2, & \dots & b_s \end{bmatrix} ; q, z $$

is defined by the power series

$$ \sum_{n=0}^{\infty} \frac{(a_1;q)_n (a_2;q)_n \dots (a_r;q)_n}{(b_1;q)_n (b_2;q)_n \dots (b_s;q)_n} \frac{(-1)^{(s-r+1)n} q^{(s-r+1)n(n-1)/2}}{(q;q)_n} z^n $$

where $z, a_1, a_2, \dots, a_r \in \mathbb{C}$, $b_1, b_2, \dots, b_s \in \mathbb{C} \setminus q^{\mathbb{Z}_-}$ and

$$ (a;q)_0 = 1, \quad (a;q)_n = (1-a)(1-aq)\dots(1-aq^{n-1}) $$

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- **Here we will need only ${}_0\phi_1$ and ${}_1\phi_1$.**

- The theta function:

$$\theta_q(z) := (z;q)_\infty (q/z;q)_\infty = \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} q^{n(n-1)/2} (-z)^n$$

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- Jacobi's theta functions:

$$\begin{aligned}
\vartheta_1(z \mid q) &= \mathrm{i} q^{1/4} e^{-\mathrm{i}z} (q^2;q^2)_\infty \, \theta_{q^2}\left(e^{2\mathrm{i}z}\right) \\
\vartheta_2(z \mid q) &= q^{1/4} e^{-\mathrm{i}z} (q^2;q^2)_\infty \, \theta_{q^2}\left(-e^{2\mathrm{i}z}\right) \\
\vartheta_3(z \mid q) &= (q^2;q^2)_\infty \, \theta_{q^2}\left(-q e^{2\mathrm{i}z}\right) \\
\vartheta_4(z \mid q) &= (q^2;q^2)_\infty \, \theta_{q^2}\left(q e^{2\mathrm{i}z}\right)
\end{aligned}$$

Contents

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- Operator A with $\mathrm{Dom}\, A = \mathrm{span}\{e_n \mid n \in \mathbb{Z}\}$ acting as

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Proposition (self-adjoint extensions)

For $t \in \mathbb{R} \cup \{\infty\}$, operators A_t, acting as $A_t\psi = \mathcal{A}\psi$, with domains
$$\operatorname{Dom} A_t = \left\{\psi \in D \mid \lim_{n\to\infty} q^{-n} (\psi_{2n+1} + t\psi_{2n}) = 0 \ \wedge \ \lim_{n\to\infty} q^{-n} (q\psi_{2n-1} - t\psi_{2n}) = 0\right\},$$
if $t \in \mathbb{R}$, or
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are all self-adjoint extensions of A.

- In addition,

$$\sigma_c(A_t) = \sigma_{\mathrm{ess}}(A_t) = \{0\}, \quad \forall t \in \mathbb{R} \cup \{\infty\}.$$

Solutions of the eigenvalue equation

- The q-exponential function:

$$\mathcal{E}_q(z) = \sum_{n=0}^{\infty} \frac{q^{n^2/4}}{(q;q)_n} z^n = {}_1\phi_1\left(0; -q^{1/2}; q^{1/2}, -q^{1/4}z\right)$$

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- Sequences $\psi^{\pm}$, where

$$\psi_n^{\pm} := (\pm\mathrm{i})^n q^{n/2} \mathcal{E}_{q^2}\left(\pm\mathrm{i}xq^n\right),$$

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- By inspection of the asymptotic behavior of $\psi_n^{\pm}$, as $n \to \pm\infty$, one gets:

$$\psi^{\pm} \in \ell^2(+\infty), \quad \text{however,} \quad \psi^{\pm} \notin \ell^2(-\infty).$$

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- Hence, one expects there are non-trivial coefficients $a = a(x)$ and $b = b(x)$ such that

$$a\psi^{+} + b\psi^{-} \in \ell^2(-\infty).$$

Proposition

For all $x \in \mathbb{C} \setminus \{0\}$, the sequence

$$\varphi(x) := \theta_q\left(-\mathrm{i}q^{-1/2}x\right)\psi^{(-)}(x) + \theta_q\left(\mathrm{i}q^{-1/2}x\right)\psi^{(+)}(x),$$

is the non-trivial solution of $\mathcal{A}\phi = x\phi$ which belongs to $\ell^2(\mathbb{Z})$.
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Moreover,

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$$\|\varphi(x)\|_{\ell^2}^2 = 4\frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2}\theta_{q^2}\left(-z^2\right)$$

Theorem (secular equation)

For $t \in \mathbb{R} \cup \{\infty\}$, one has $\mathrm{spec}_c(A_t) = \{0\}$ and $\mathrm{spec}_p(A_t)$ coincides with the set of roots of the secular equation:

$$x\theta_{q^4}\left(q^2x^2\right) + t\theta_{q^4}\left(x^2\right) = 0, \quad \text{for } t \in \mathbb{R},$$

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Corollary:

$$\mathrm{spec}_p(A_0) = \left\{\pm q^{2n+1} \mid n \in \mathbb{Z}\right\} \quad \text{and} \quad \mathrm{spec}_p(A_\infty) = \left\{\pm q^{2n} \mid n \in \mathbb{Z}\right\}.$$

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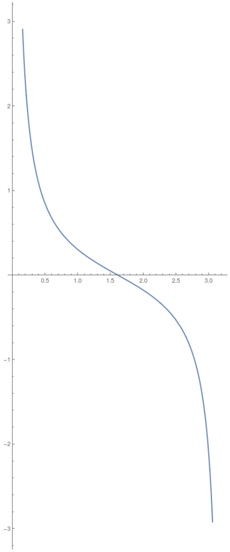
- How to solve the secular equation in general?
- Reparametrize $t = \Phi(s)$ and use nice properties of Jacobi's theta functions...

- The function

$$\Phi(s) := \mathrm{i}q^{1/2}\frac{\vartheta_4\left(\mathrm{i}s \mid q^2\right)}{\vartheta_1\left(\mathrm{i}s \mid q^2\right)}$$

is real-valued, strictly decreasing on $(0, -2\ln q)$, and maps $[0, -2\ln q)$ onto $\mathbb{R} \cup \{\infty\}$.



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- For the inverse function, one has

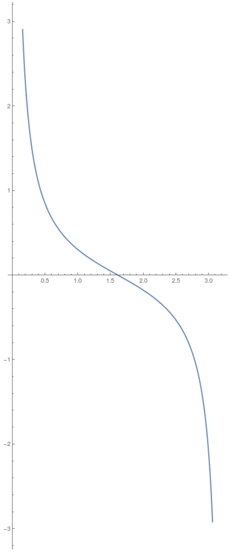
$$\Phi^{-1}(t) = C(q) \int_t^{\infty} \frac{\mathrm{d}x}{\sqrt{\left(D(q)+x^2\right)\left(q+x^2\right)}}$$

where

$$C(q) = \frac{q^{1/2}}{\vartheta_2\left(0 \mid q^2\right)\vartheta_3\left(0 \mid q^2\right)}$$

and

$$D(q) = \frac{q\vartheta_3^2\left(0 \mid q^2\right)}{\vartheta_2^2\left(0 \mid q^2\right)}.$$



Spectrum fully explicitly

- Using the reparametrization $t = \Phi(s)$, the secular equation simplifies to

$$\theta_{q^2}\left(e^{-s}x\right)\theta_{q^2}\left(-e^{s}x\right) = 0.$$

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Theorem

Let $t \in \mathbb{R} \cup \{\infty\}$, then

$$\mathrm{spec}_p(A_t) = -e^{-s}q^{2\mathbb{Z}} \cup e^{s}q^{2\mathbb{Z}}$$

where

$$s = C(q)\int_t^{\infty} \frac{\mathrm{d}x}{\sqrt{\left(D(q)+x^2\right)\left(q+x^2\right)}}.$$

In addition, the family of corresponding eigenvectors $\{\varphi(\pm e^{s}q^{2N}) \mid N \in \mathbb{Z}\}$, where

$$\varphi_n(x) = (-1;q)_\infty\, x^n q^{n(n-1)/2}\, {}_0\phi_1\left(-;0;q^2,q^{-2n+4}x^{-2}\right),$$

forms an orthogonal basis of $\ell^2(\mathbb{Z})$.

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The second Jacobi matrix $\mathcal{B}$ determines the unique operator

$$B = U + U^* + \alpha V$$

where U is the forward shift operator and V is the self-adjoint diagonal operator:

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Proposition (essential spectrum)

The operator B is self-adjoint and one has

$$\sigma_{\text{ess}}(B) = [-2, 2].$$

- Using properties of the Hahn-Exton q-Bessel functions one verifies:

$$f_n(z) := (-1)^n \alpha^{-n} q^{\frac{1}{2}n(n+1)} \left(z^{-1}\alpha^{-1}q^{n+1}; q\right)_\infty \, {}_1\phi_1\left(0; z^{-1}\alpha^{-1}q^{n+1}; q, z\alpha^{-1}q^{n+1}\right)$$

and

$$g_n(z) := z^{-n}\left(z\alpha q^{1-n}; q\right)_\infty \, {}_1\phi_1\left(0; z\alpha q^{1-n}; q, qz^2\right).$$

are two solutions of the equation

$$\mathcal{B}\psi = (z + z^{-1})\psi.$$

for all $\alpha, z \neq 0$.

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- Note that

$$z \mapsto z + z^{-1} : \begin{cases} \{z \mid 0<|z|<1\} \to \mathbb{C}\setminus[-2,2], & \text{(outside } \sigma_{\text{ess}}(A)\text{)} \\ \{e^{i\theta} \mid \theta\in[0,\pi]\} \to [-2,2], & \text{(inside } \sigma_{\text{ess}}(A)\text{)} \end{cases}$$

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$$z \mapsto z + z^{-1} : \begin{cases} \{z \mid 0 < |z| < 1\} \to \mathbb{C} \setminus [-2, 2], & \text{(outside } \sigma_{\text{ess}}(A)\text{)} \\ \{e^{i\theta} \mid \theta \in [0, \pi]\} \to [-2, 2], & \text{(inside } \sigma_{\text{ess}}(A)\text{)} \end{cases}$$

- The solutions $f(z)$ and $g(z)$ are linearly independent iff $z \notin \alpha^{-1}q^{\mathbb{Z}} \cup \{0\}$ since

$$W(f, g) = -z^{-1}\theta_q(\alpha z).$$

Detailed asymptotic analysis of solutions f and g yields:

- If $0 < |z| < 1$ and $z \notin \alpha^{-1}q^{\mathbb{Z}} \cup \{0\}$, then

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- If $|z| = 1$ the asymptotic behavior of solutions is very different and, in the end, it implies that: *For* $\forall \alpha \in \mathbb{R}$ *and* $\forall x \in [-2, 2]$*, there is no non-trivial solution of* $\mathcal{B}\psi = x\psi$ *belonging to* $\ell^2(\mathbb{Z})$*.*

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Theorem (point spectrum)

If $\alpha \neq 0$, then
$$\sigma(B) \setminus [-2, 2] = \sigma_p(B) = \left\{ \alpha^{-1} q^m + \alpha q^{-m} \mid m > \lfloor \log_q |\alpha| \rfloor \right\}$$
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In addition,

$$\|\mathbf{v}_m\|_{\ell^2(\mathbb{Z})} = \frac{|\alpha|^{-m} q^{m(m+1)/2}}{\sqrt{1 - \alpha^{-2} q^{2m}}} (q; q)_\infty, \quad m > \lfloor \log_q |\alpha| \rfloor.$$

The absolutely continuous part of the spectral measure

- Let us denote

$$E_{k,l}(\cdot) := \langle e_k, E_B(\cdot)e_l \rangle, \quad k, l \in \mathbb{Z},$$

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$$E_{k,l}((a,b)) = \lim_{\delta\to 0+} \lim_{\epsilon\to 0+} \frac{1}{2\pi\mathrm{i}} \int_{a+\delta}^{b-\delta} \left(G_{k,l}(x+\mathrm{i}\epsilon) - G_{k,l}(x-\mathrm{i}\epsilon) \right) \mathrm{d}x,$$
where
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Proposition

Let $\alpha \neq 0$ and $-2 \leq a < b \leq 2$. Then for any $k, l \in \mathbb{Z}$, it holds
$$E_{k,l}([a,b]) = \frac{1}{2\pi} \int_{\phi_b}^{\phi_a} f_l\left(e^{\mathrm{i}\phi}\right) f_k\left(e^{\mathrm{i}\phi}\right) \left| \frac{\left(e^{2\mathrm{i}\phi}; q\right)_\infty}{\left(\alpha e^{\mathrm{i}\phi}, q\alpha^{-1} e^{-\mathrm{i}\phi}; q\right)_\infty} \right|^2 \mathrm{d}\phi$$
where $\phi_a = \arccos(a/2)$ and $\phi_b = \arccos(b/2)$. Consequently, $\sigma_{ac}(B) = [-2, 2]$.

Theorem

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In addition, for $\mathcal{M} \subset \mathbb{R}$ a Borel set, we have

$$E_{k,l}(\mathcal{M}) = \frac{1}{2\pi} \int_{2\cos\phi \in [-2,2] \cap \mathcal{A}} f_l\left(e^{\mathrm{i}\phi}\right) f_k\left(e^{\mathrm{i}\phi}\right) \left| \frac{\left(e^{2\mathrm{i}\phi}; q\right)_\infty}{\left(\alpha e^{\mathrm{i}\phi}, q\alpha^{-1} e^{-\mathrm{i}\phi}; q\right)_\infty} \right|^2 \mathrm{d}\phi$$
$$+ \frac{1}{(q;q)_\infty^2} \sum_{\substack{m > \lfloor \log |\alpha| \rfloor \\ \alpha^{-1} q^m + \alpha q^{-m} \in \mathcal{M}}} \left(1 - \alpha^{-2} q^{2m}\right) \alpha^{2m} q^{-m(m+1)} f_l\left(\alpha^{-1} q^m\right) f_k\left(\alpha^{-1} q^m\right).$$

One consequence for special functions

- Recall the Hanhn-Exton (or third Jackson's) q-Bessel function is defined as

$$J_{\nu}(z;q) = z^{\nu} \frac{(q^{\nu+1};q)_{\infty}}{(q;q)_{\infty}} {}_{1}\phi_{1}\left(0;q^{\nu+1};q,qz^{2}\right).$$

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- Elements of the eigenvectors $\boldsymbol{v}_m$ are expressible in terms of Hanhn-Exton q-Bessel function and the formula for the norm of the eigenvectors yields

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- **This formula seems to be new (Really?)** and it generalizes the well-known summation formula for the Bessel functions of the first kind:

$$\sum_{n\in\mathbb{Z}} J_n^2(z) = 1, \quad |z|<1.$$

Thank you!