On the Eigenvalue Problem for a Particular Class of Jacobi Matrices

František Štampach, Pavel Šťovíček[1]

[1]Czech Technical University in Prague

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Outline

Functions $\mathfrak{E}$, $\mathfrak{F}$

Definition

Define $\mathfrak{E} : D \to \mathbb{C}$, $\mathfrak{F} : D \to \mathbb{C}$

$$\mathfrak{E}(x) = 1 + \sum_{m=1}^{\infty} \sum_{k_1=1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \cdots \sum_{k_m=k_{m-1}+2}^{\infty} x_{k_1} x_{k_1+1} x_{k_2} x_{k_2+1} \dots x_{k_m} x_{k_m+1}$$

and

$$\mathfrak{F}(x) = 1 + \sum_{m=1}^{\infty} (-1)^m \sum_{k_1=1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \cdots \sum_{k_m=k_{m-1}+2}^{\infty} x_{k_1} x_{k_1+1} x_{k_2} x_{k_2+1} \dots x_{k_m} x_{k_m+1}$$

where

$$D = \left\{ \{x_k\}_{k=1}^{\infty} \subset \mathbb{C};\ \sum_{k=1}^{\infty} |x_k x_{k+1}| < \infty \right\}.$$

For a finite number of complex variables we identify $\mathfrak{F}(x_1, x_2, \dots, x_n)$ with $\mathfrak{F}(x)$ where $x = (x_1, x_2, \dots, x_n, 0, 0, 0, \dots)$ and similarly for $\mathfrak{E}$.

Properties of the functions $\mathfrak{E}$ and $\mathfrak{F}$

- The domain D is not a linear space. One has, however, $\ell^2(\mathbb{N}) \subset D$.
- $\mathfrak{E}$ and $\mathfrak{F}$ are continuous functionals on $\ell^2(\mathbb{N})$.
- For all $x \in D$ and $k = 1, 2, \ldots$ reccurence relations

Reccurence relations

$$\mathfrak{F}(x) = \mathfrak{F}(x_1, \ldots, x_k)\,\mathfrak{F}(T^k x) - \mathfrak{F}(x_1, \ldots, x_{k-1}) x_k x_{k+1} \mathfrak{F}(T^{k+1} x),$$

$$\mathfrak{E}(x) = \mathfrak{E}(x_1, \ldots, x_k)\,\mathfrak{E}(T^k x) + \mathfrak{E}(x_1, \ldots, x_{k-1}) x_k x_{k+1} \mathfrak{E}(T^{k+1} x)$$

hold. T is the shift operator, i.e. $T^k \{x_n\}_{n=1}^{\infty} = \{x_n\}_{n=k+1}^{\infty}$.

Examples - geometric sequence

Let $t, w \in \mathbb{C}$, $|t| < 1$ then it holds

$$\mathfrak{F}\left(\{t^{k-1}w\}_{k=1}^{\infty}\right) = 1 + \sum_{m=1}^{\infty}(-1)^m \frac{t^{m(2m-1)}w^{2m}}{(1-t^2)(1-t^4)\ldots(1-t^{2m})},$$

$$\mathfrak{C}\left(\{t^{k-1}w\}_{k=1}^{\infty}\right) = 1 + \sum_{m=1}^{\infty} \frac{t^{m(2m-1)}w^{2m}}{(1-t^2)(1-t^4)\ldots(1-t^{2m})}.$$

- The series on the RHS can be identify with a basic hypergeometric series.

Examples - Bessel functions

Let $w \in \mathbb{C}$ and $\nu \notin -\mathbb{N}$ then it holds

$$J_\nu(2w) = \frac{w^\nu}{\Gamma(\nu+1)} \mathfrak{F}\left(\left\{\frac{w}{\nu+k}\right\}_{k=1}^{\infty}\right),$$

$$I_\nu(2w) = \frac{w^\nu}{\Gamma(\nu+1)} \mathfrak{C}\left(\left\{\frac{w}{\nu+k}\right\}_{k=1}^{\infty}\right).$$

The reccursive relation for $\mathfrak{F}$ and $\mathfrak{C}$ transforms to the well known identities

- $wJ_{\nu-1}(2w) - \nu J_\nu(2w) + wJ_{\nu+1}(2w) = 0,$
- $wI_{\nu-1}(2w) - \nu I_\nu(2w) - wI_{\nu+1}(2w) = 0.$

Jacobi matrices of a special type

- Let $\{w_n\}_{n=1}^{\infty}$ is positive and bouded sequence.
- Let $\{\lambda_n\}_{n=1}^{\infty}$ is real strictly increasing sequence satisfying the condition

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} < \infty.$$

- Denote

$$J := \begin{pmatrix} \lambda_1 & w_1 & & & \\ w_1 & \lambda_2 & w_2 & & \\ & w_2 & \lambda_3 & w_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

$$J_n := \begin{pmatrix} \lambda_1 & w_1 & & & \\ w_1 & \lambda_2 & w_2 & & \\ & \ddots & \ddots & \ddots & \\ & & w_{n-2} & \lambda_{n-1} & w_{n-1} \\ & & & w_{n-1} & \lambda_n \end{pmatrix}.$$

General results

Proposition

J regarded as an operator in $\ell^2(\mathbb{N})$ is self-adjoint. Any eigenvalue of J is simple. The spectrum of J is discrete.

Application of $\mathfrak{F}$ to a finite complex sequence can be unambiguously defined with the aid of a determinant of a Jacobi matrix:

For $d \in \mathbb{N}$ and $\{x_j\}_{j=1}^d \subset \mathbb{C}$, one has

$$\mathfrak{F}(x_1, x_2, \ldots, x_d) = \left| \begin{pmatrix} 1 & x_1 & & & & \\ x_2 & 1 & x_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & x_{d-1} & 1 & x_{d-1} \\ & & & & x_d & 1 \end{pmatrix} \right| .$$

Decomposition

The Jacobi matrix J_n can be decomposed into the product

$$J_n = G_n \tilde{J}_n G_n$$

where $G_n = \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_n)$ and

$$\tilde{J}_n = \begin{pmatrix} \tilde{\lambda}_1 & 1 & & & & \\ 1 & \tilde{\lambda}_2 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & \tilde{\lambda}_{n-1} & 1 \\ & & & & 1 & \tilde{\lambda}_n \end{pmatrix}.$$

One can put $\tilde{\lambda}_k = \lambda_k/\gamma_k^2$ and

$$\gamma_{2k-1} = \prod_{j=1}^{k-1} \frac{w_{2j}}{w_{2j-1}}, \ \gamma_{2k} = w_1 \prod_{j=1}^{k-1} \frac{w_{2j+1}}{w_{2j}}, \ k = 1, 2, 3, \ldots.$$

Alternatively, $\gamma_1 = 1$, $\gamma_{k+1} = w_k/\gamma_k$.

Characteristic function

The characteristic function of finite symmetric Jacobi matrix J_n can be expressed with the aid of $\mathfrak{F}$:

Let $n \in \mathbb{N}$ and $z \in \mathbb{C}$ then it holds

$$\det(J_n - zI_n) = \left(\prod_{k=1}^{n}(\lambda_n - z)\right)\mathfrak{F}\left(\frac{\gamma_1^2}{\lambda_1 - z}, \frac{\gamma_2^2}{\lambda_2 - z}, \dots, \frac{\gamma_n^2}{\lambda_n - z}\right).$$

- What one can say about function $\mathfrak{F}\left(\left\{\frac{\gamma_k^2}{\lambda_k - z}\right\}_{k=1}^{\infty}\right)$?
- Is this function related to the spectrum of J in some way?

Lemma

It holds

$$\mathfrak{F}\left(\left\{\frac{\gamma_k^2}{\lambda_k - z}\right\}_{k=1}^{n}\right) \overset{n\to\infty}{\rightrightarrows} \mathfrak{F}\left(\left\{\frac{\gamma_k^2}{\lambda_k - z}\right\}_{k=1}^{\infty}\right) \text{ locally in } z \in \mathbb{C} \setminus \{\lambda_k\},$$

$$\frac{d}{dz}\mathfrak{F}\left(\left\{\frac{\gamma_k^2}{\lambda_k - z}\right\}_{k=1}^{n}\right) \overset{n\to\infty}{\rightrightarrows} \frac{d}{dz}\mathfrak{F}\left(\left\{\frac{\gamma_k^2}{\lambda_k - z}\right\}_{k=1}^{\infty}\right) \text{ locally in } z \in \mathbb{C} \setminus \{\lambda_k\}.$$

Properties of $\mathfrak{F}\left(\left\{\frac{\gamma_k^2}{\lambda_k - z}\right\}_{k=1}^{\infty}\right)$:

- The function is holomorphic on $\mathbb{C} \setminus \{\lambda_k\}$.
- The function has poles in points $z \in \{\lambda_k\}$ of order 1.

Limit points.

- By E. K. Ifantis, C. G. Kokologiannaki, E. Petropoulou: *Limit points of eigenvalues of truncated unbounded tridiagonal operators*, Centr. Europ. J. Math. (2007) 335-344. the equality

$$\mathrm{spec}(J) = \Lambda(J)$$

holds. $\Lambda(J)$ is the set of all points which are limit points of eigenvalues of J_n when $n \to \infty$.
- In other words one has the equivalence $\lambda \in \mathrm{spec}(J)$ if and only if

$$(\exists\{k_n\} \subset \mathbb{N}, k_n < k_{n+1})(\exists\{\tilde{\lambda}_n\} \subset \mathbb{R}, \tilde{\lambda}_n \in \mathrm{spec}(J_{k_n}))(\lim_{n\to\infty} \tilde{\lambda}_n = \lambda).$$

First implication

The last equivalence together with the statement refer to local uniform convergent is fundamental to prove the following proposition:

Proposition

The implication

$$(\lambda \in \operatorname{spec}(J) \wedge \lambda \notin \{\lambda_k\}) \implies \mathfrak{F}\left(\left\{\frac{\gamma_k^2}{\lambda_k - \lambda}\right\}_{k=1}^{\infty}\right) = 0$$

holds.

Christoffel-Darboux-like identity

Since identity

$$\sum_{k=1}^{n}\left(\prod_{l=2}^{k}\left(\frac{z-\lambda_l}{w_{l-1}}\right)\mathfrak{F}\left(\left\{\frac{\gamma_l^2}{\lambda_l-z}\right\}_{l=k+1}^{n}\right)\right)^2 = \mathfrak{F}\left(\left\{\frac{\gamma_l^2}{\lambda_l-z}\right\}_{l=1}^{n}\right)\mathfrak{F}\left(\left\{\frac{\gamma_l^2}{\lambda_l-z}\right\}_{l=2}^{n}\right)$$
$$+(z-\lambda_1)\left[\mathfrak{F}\left(\left\{\frac{\gamma_l^2}{\lambda_l-z}\right\}_{l=2}^{n}\right)\frac{d}{dz}\mathfrak{F}\left(\left\{\frac{\gamma_l^2}{\lambda_l-z}\right\}_{l=1}^{n}\right)-\mathfrak{F}\left(\left\{\frac{\gamma_l^2}{\lambda_l-z}\right\}_{l=1}^{n}\right)\frac{d}{dz}\mathfrak{F}\left(\left\{\frac{\gamma_l^2}{\lambda_l-z}\right\}_{l=2}^{n}\right)\right]$$

holds one can prove the following statement.

Proposition

Zeros of the function $\mathfrak{F}\left(\left\{\frac{\gamma_k^2}{\lambda_k-z}\right\}_{k=1}^{\infty}\right)$ are simple.

Second implication

Proposition

Let $z_0 \in \mathbb{C} \setminus \{\lambda_k\}$ such that

$$\mathfrak{F}\left(\left\{\frac{\gamma_k^2}{\lambda_k - z_0}\right\}_{k=1}^{\infty}\right) = 0$$

then $z_0 \in \operatorname{spec}(J)$.

- We have shown the equality

$$\operatorname{spec}(J) \setminus \{\lambda_k\} = \left\{z \in \mathbb{C};\quad \mathfrak{F}\left(\left\{\frac{\gamma_k^2}{\lambda_k - z}\right\}_{k=1}^{\infty}\right) = 0\right\}$$

- What about the points $z \in \{\lambda_k\}$?

A similar relation for the points $\{\lambda_k\}_{k=1}^{\infty}$ was found:

Proposition

Let $s \in \mathbb{N}$ then $\lambda_s \in \operatorname{spec}(J)$ if and only if

$$\lim_{z \to \lambda_s} (\lambda_s - z)\mathfrak{F}\left(\left\{\frac{\gamma_k^2}{\lambda_k - z}\right\}_{k=1}^{\infty}\right) = 0.$$

To derive this result it is necessary:

- Prove the statement concerning the local uniform convergence.
- Find an analogy of Christoffel-Darboux identity for the function under investigation.
- Prove if λ_s is a zero of the function then it is simple.

Summary

Let $\{\lambda_n\}_{n=1}^{\infty}$ be real and strictly increasing seq. satisfying

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} < \infty$$

and $\{w_n\}_{n=1}^{\infty}$ be bounded and positive seq. then it holds

1. $$z \in \operatorname{spec}(J) \setminus \{\lambda_n\}_{n=1}^{\infty} \iff \mathfrak{F}\left(\left\{\frac{\gamma_k^2}{\lambda_k - z}\right\}_{k=1}^{\infty}\right) = 0,$$

2. $$\lambda_s \in \operatorname{spec}(J) \iff \lim_{z\to\lambda_s} (\lambda_s - z)\mathfrak{F}\left(\left\{\frac{\gamma_k^2}{\lambda_k - z}\right\}_{k=1}^{\infty}\right) = 0.$$

Example

- Let $\lambda_n = n$ and $w_n = w > 0$ for all $n \in \mathbb{N}$. Then

$$J = \begin{pmatrix} 1 & w & & & \\ w & 2 & w & & \\ & w & 3 & w & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

- With this choice it holds

$$\gamma_n = \begin{cases} 1, & \text{if } n \text{ is odd} \\ w, & \text{if } n \text{ is even} \end{cases}$$

- The "characteristic" function can be expressed as

$$\mathfrak{F}\left(\left\{\frac{\gamma_k^2}{k-z}\right\}_{k=1}^{\infty}\right) = \mathfrak{F}\left(\left\{\frac{w}{k-z}\right\}_{k=1}^{\infty}\right) = w^z \Gamma(1-z) J_{-z}(2w).$$

Example

Since term $w^z\Gamma(1-z)$ does not effect the zeros of function $\mathfrak{F}\left(\left\{\frac{w}{k-z}\right\}_{k=1}^{\infty}\right)$ and moreover the term $\Gamma(1-z)$ makes the singularities in $z=1,2,\ldots$ of the function $\mathfrak{F}\left(\left\{\frac{w}{k-z}\right\}_{k=1}^{\infty}\right)$ one arrives at the statement

$$z \in \operatorname{spec}(J) \iff J_{-z}(2w) = 0,$$

or equivalently

$$\operatorname{spec}(J) = \{z \in \mathbb{C};\ J_{-z}(2w) = 0\}.$$