Non-self-adjoint Toeplitz matrices with purely real spectrum and related problems

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Based on: B. Shapiro, F. Štampach: `Non-self-adjoint Toeplitz matrices whose principal submatrices have real spectrum`, arXiv:1702.00741 [math.CA]

Contents

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Remark:
If a is analytic in $\mathbb{C} \setminus \{0\}$ (especially, if a is a Laurent polynomial), then the assumption 1 can be omitted.

The case of banded Toeplitz matrices

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Remark:

It is a very surprising feature of banded Toeplitz matrices that the asymptotic reality of the eigenvalues (claim 1) forces all eigenvalues of all $T_n(b)$ to be real (claim 3). Hence, if, for instance, the 2×2 matrix $T_2(b)$ has a non-real eigenvalue, there is no chance for the limiting set $\Lambda(b)$ to be real!

Examples

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$$b(z) = z^{-1} + az, \qquad (a \in \mathbb{C} \setminus \{0\}).$$

Then

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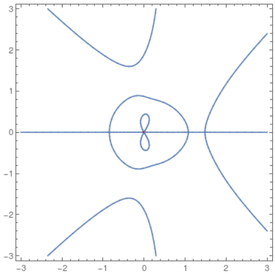

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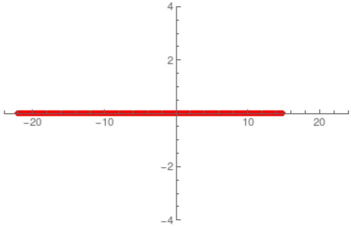

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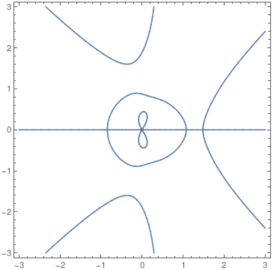

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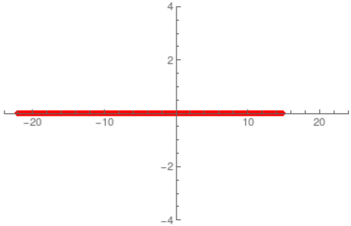

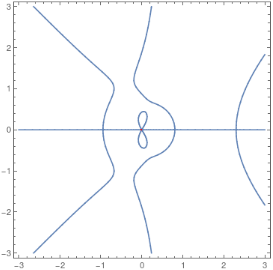

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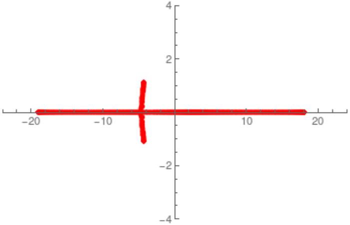

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- The set $\Lambda(b)$ can be described in terms of zeros of the polynomial $z \mapsto z^r(b(z) - \lambda)$ [Schmidt and Spitzer, 1960].
- The weak limit of the eigenvalue-counting measures of $T_n(b)$:

$$\mu_n = \frac{1}{n} \sum_{k=1}^{n} \delta_{\lambda_k^{(n)}}$$

exists, as $n \to \infty$, and is absolutely continuous measure μ supported on $\Lambda(b)$ whose density can be expressed in terms of zeros of $z \mapsto z^r(b(z) - \lambda)$ [Hirschman Jr., 1967].

The limiting measure and the Jordan curve without critical points

1. Let $T_n(b)$ be a banded Toeplitz matrix with **real** elements:

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Theorem:

Let $b'(\gamma(t)) \neq 0$ for all $t \in (0, \pi)$. Then $b \circ \gamma$ restricted to $(0, \pi)$ is either strictly increasing or decreasing; the limiting measure μ is supported on the interval $[\alpha, \beta] := b(\gamma([0, \pi]))$ and its density satisfies

$$\frac{\mathrm{d}\mu}{\mathrm{d}x}(x) = \pm\frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}x}(b \circ \gamma)^{-1}(x),$$

for $x \in (\alpha, \beta)$, where the $+$ sign is used when $b \circ \gamma$ increases on $(0, \pi)$, and the $-$ sign is used otherwise.

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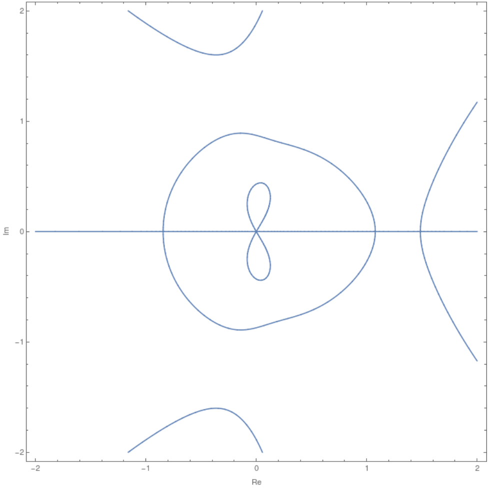

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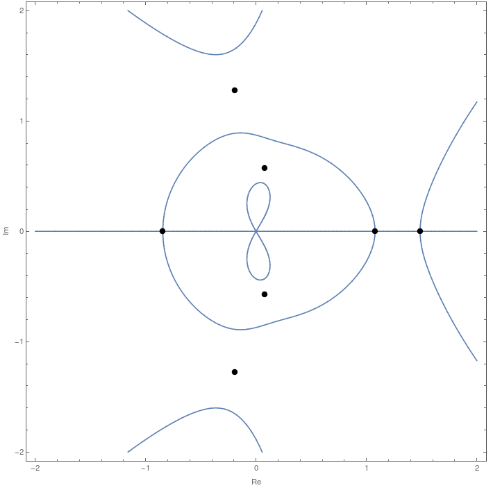

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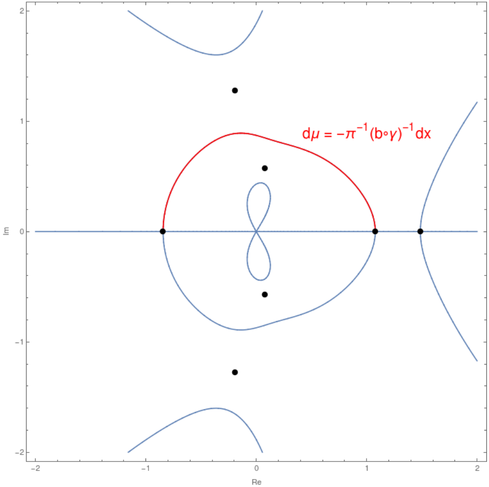

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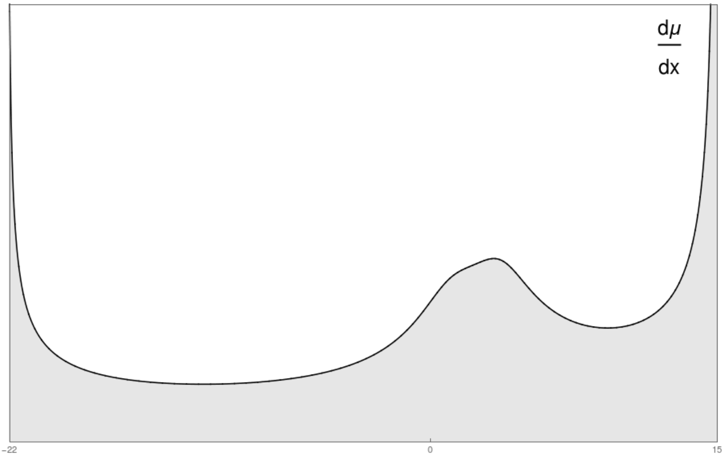

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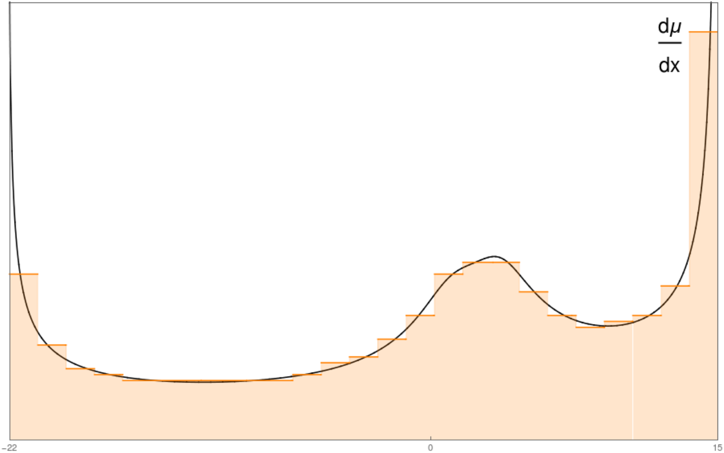

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Theorem:

Suppose that b and γ are as before and let $\ell \in \mathbb{N}_0$ be the number of critical points of b in $\gamma((0,\pi))$ and $0 =: \phi_0 < \phi_1 < \cdots < \phi_\ell < \phi_{\ell+1} := \pi$ are such that $b'(\gamma(\phi_j)) = 0$ for all $0 \leq j \leq \ell+1$. Then $b \circ \gamma$ restricted to (ϕ_{i-1}, ϕ_i) is strictly monotone for all $1 \leq i \leq \ell+1$, and the limiting measure $\mu = \mu_1 + \mu_2 + \cdots + \mu_{\ell+1}$, where μ_i is an absolutely continuous measure supported on $[\alpha_i, \beta_i] := b(\gamma([\phi_{i-1}, \phi_i]))$ whose density is given by

$$\frac{\mathrm{d}\mu_i}{\mathrm{d}x}(x) = \pm\frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}x}(b\circ\gamma)^{-1}(x)$$

for all $x \in (\alpha_i, \beta_i)$ and all $i \in \{1, 2, \ldots, \ell+1\}$. The $+$ sign is used when $b \circ \gamma$ increases on (α_i, β_i), and the $-$ sign is used otherwise.

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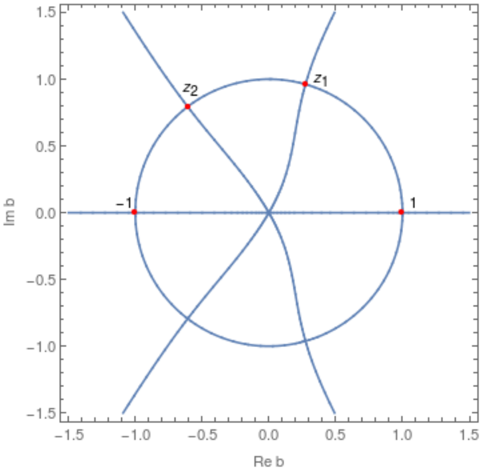

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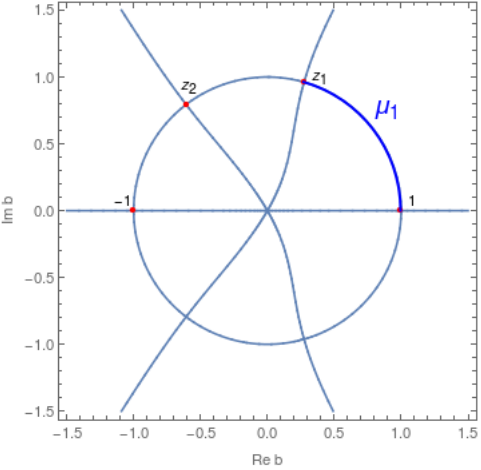

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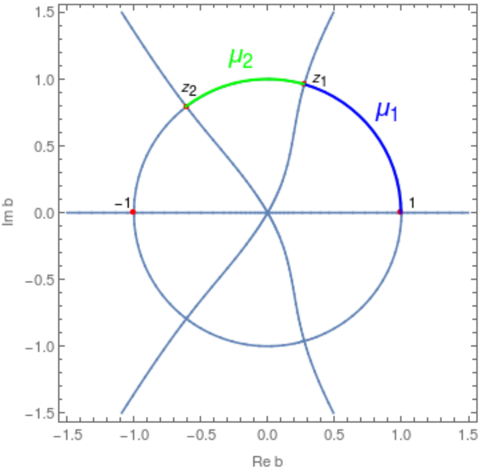

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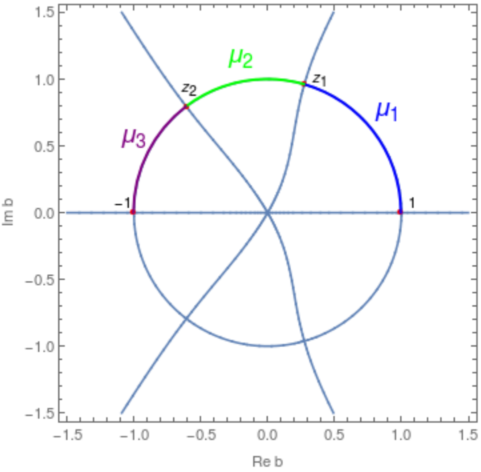

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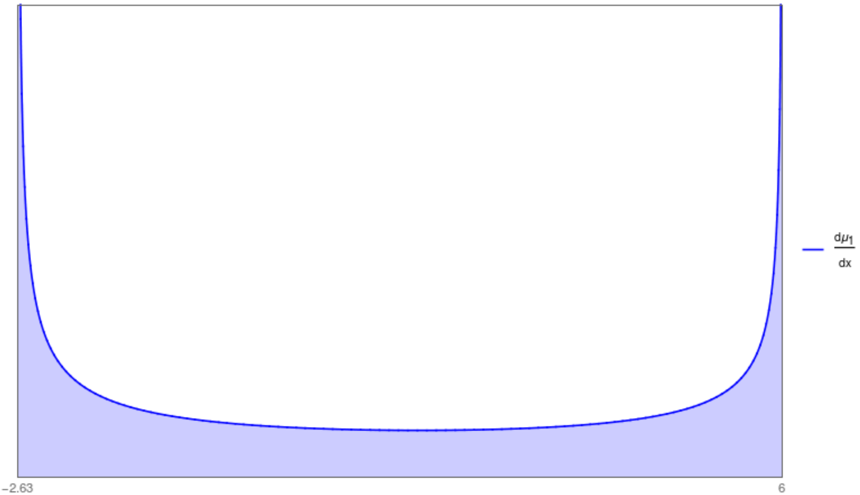

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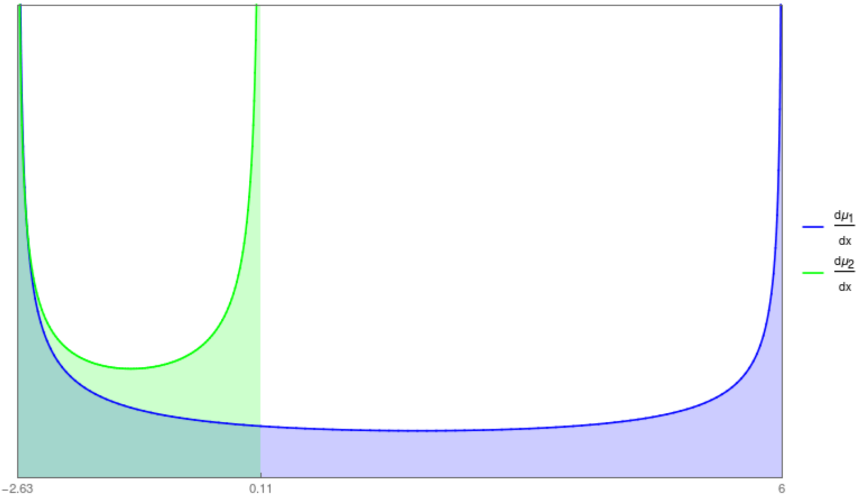

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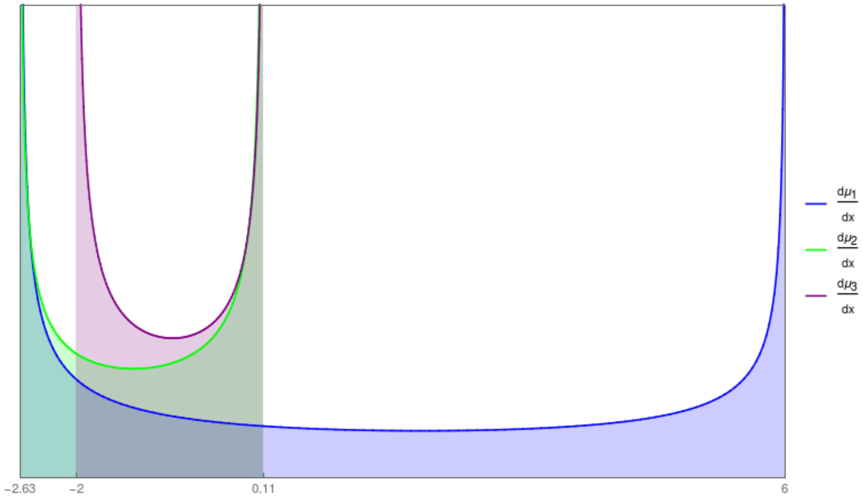

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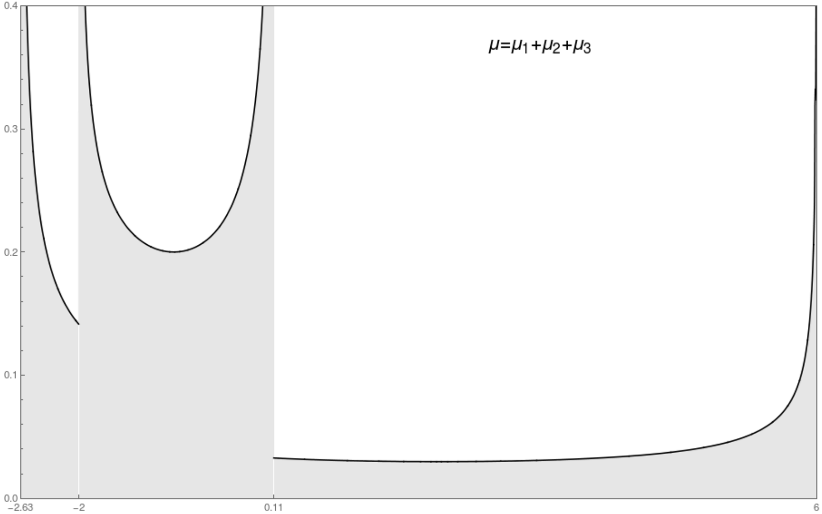

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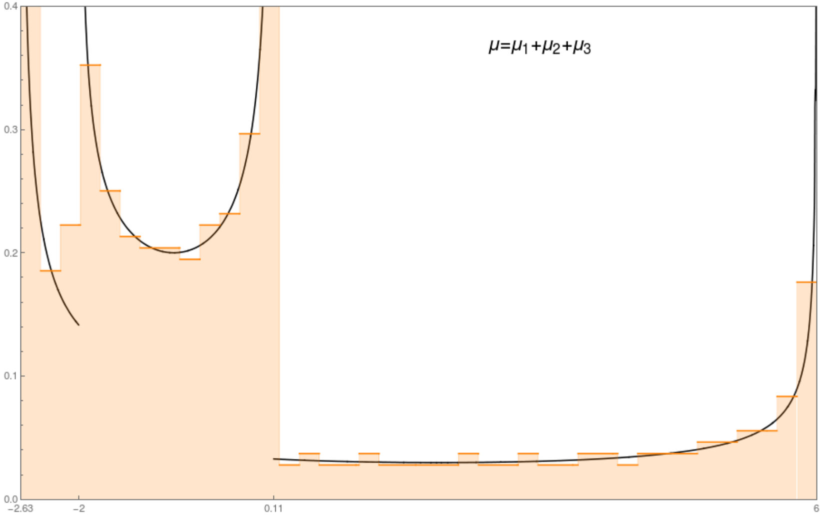

Contents

The limiting measure as a solution of the HMP

- We consider real Laurent polynomial symbols:

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(If a counter-example exists, $\mathbb{C} \setminus \Lambda(b)$ has to be disconnected.)

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2. the OGPs $\{p_n\}$ belong to the Blumenthal–Nevai class $M((\beta-\alpha)/2, (\alpha+\beta)/2)$, i.e.,

$$\lim_{n\to\infty} a_n = \frac{\beta-\alpha}{4} \quad \text{and} \quad \lim_{n\to\infty} b_n = \frac{\alpha+\beta}{2}.$$

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$$b(z) = \frac{1}{z^r}(1+az)^{r+s}, \qquad (a>0, r, s \in \mathbb{N}).$$

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- The limiting measure μ is the solution of the moment problem with moments

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- Explicit formulas for the Jacobi parameters a_n and b_n are not known in general but we have

$$2\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \frac{(r+s)^{r+s}}{2r^r s^s}.$$

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- Jacobi parameters:

$$a_1^2 = 6a^2, \quad a_k^2 = \frac{9(6k-5)(6k-1)(3k-1)(3k+1)}{4(4k-3)(4k-1)^2(4k+1)}a^2, \quad \text{for } k > 1.$$

  and

$$b_1 = 3a, \quad b_k = \frac{3(36k^2 - 54k + 13)}{2(4k-5)(4k-1)}a, \quad \text{for } k > 1.$$

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$$r_n^{(\alpha,\beta)}(x;c) := \frac{2^n(c+\alpha+\beta+1)_n(c+1)_n}{(2c+\alpha+\beta+1)_{2n}} P_n^{(\alpha,\beta)}(2x-1;c), \quad n \in \mathbb{N}_0,$$

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- This relation and the known properties of the associated Jacobi polynomials allow to derive other formulas for p_n such as: an explicit representation, a generating function, ...

Thank you!