# On the eigenvalue problem for a certain class of infinite Jacobi matrices

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# Outline

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   - Two examples
2. **Symmetric Jacobi matrices**
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## Definition

Let me define $\mathfrak{E} : D \to \mathbb{C}$ a $\mathfrak{F} : D \to \mathbb{C}$ by relations

$$\mathfrak{E}(x) = 1 + \sum_{m=1}^{\infty} \sum_{k_1=1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \ldots \sum_{k_m=k_{m-1}+2}^{\infty} x_{k_1} x_{k_1+1} x_{k_2} x_{k_2+1} \ldots x_{k_m} x_{k_m+1}$$

and

$$\mathfrak{F}(x) = 1 + \sum_{m=1}^{\infty} (-1)^m \sum_{k_1=1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \ldots \sum_{k_m=k_{m-1}+2}^{\infty} x_{k_1} x_{k_1+1} x_{k_2} x_{k_2+1} \ldots x_{k_m} x_{k_m+1},$$

where

$$D = \left\{ \{x_k\}_{k=1}^{\infty} \subset \mathbb{C};\ \sum_{k=1}^{\infty} |x_k x_{k+1}| < \infty \right\}.$$

For a finite number of complex variables let me identify $\mathfrak{F}(x_1, x_2, \ldots, x_n)$ with $\mathfrak{F}(x)$ where $x = (x_1, x_2, \ldots, x_n, 0, 0, 0, \ldots)$ and similarly for $\mathfrak{E}$.

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- For all $x \in D$ and $k = 1, 2, \ldots$ one has

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**Recursive relations**

$$\mathfrak{F}(x) = \mathfrak{F}(x_1, \ldots, x_k)\,\mathfrak{F}(T^k x) - \mathfrak{F}(x_1, \ldots, x_{k-1}) x_k x_{k+1} \mathfrak{F}(T^{k+1} x),$$
$$\mathfrak{E}(x) = \mathfrak{E}(x_1, \ldots, x_k)\,\mathfrak{E}(T^k x) + \mathfrak{E}(x_1, \ldots, x_{k-1}) x_k x_{k+1} \mathfrak{E}(T^{k+1} x),$$

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- Especially for $k = 1$, one gets simple relations

$$\mathfrak{F}(x) = \mathfrak{F}(Tx) - x_1 x_2\,\mathfrak{F}(T^2 x),$$

$$\mathfrak{E}(x) = \mathfrak{E}(Tx) + x_1 x_2\,\mathfrak{E}(T^2 x).$$

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2. The case of Bessel functions:

Let $w \in \mathbb{C}$ a $\nu \notin -\mathbb{N}$, then it holds

$$J_\nu(2w) = \frac{w^\nu}{\Gamma(\nu+1)}\mathfrak{F}\left(\left\{\frac{w}{\nu+k}\right\}_{k=1}^{\infty}\right), \qquad I_\nu(2w) = \frac{w^\nu}{\Gamma(\nu+1)}\mathfrak{E}\left(\left\{\frac{w}{\nu+k}\right\}_{k=1}^{\infty}\right).$$

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- Recursive relations for $\mathfrak{F}$ and $\mathfrak{E}$ written in this special case has the form:

$$\begin{aligned} wJ_{\nu-1}(2w) - \nu J_\nu(2w) + wJ_{\nu+1}(2w) &= 0, \\ wI_{\nu-1}(2w) - \nu I_\nu(2w) - wI_{\nu+1}(2w) &= 0. \end{aligned}$$

- Let positive sequence $\{w_n\}_{n=1}^{\infty}$ and real sequence $\{\lambda_n\}_{n=1}^{\infty}$ to be given.
- Let me denote

$$J := \begin{pmatrix} \lambda_1 & w_1 & & & \\ w_1 & \lambda_2 & w_2 & & \\ & w_2 & \lambda_3 & w_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

- Let $J_n$ be the $n$-th truncation of $J$, i.e. $J_n = (P_n J P_n) \upharpoonright \mathrm{Ran}\, P_n$, where $P_n$ is the orthogonal projection on the space spanned by $\{e_1, e_2, \dots, e_n\}$. In other words,

$$J_n = \begin{pmatrix} \lambda_1 & w_1 & & & \\ w_1 & \lambda_2 & w_2 & & \\ & \ddots & \ddots & \ddots & \\ & & w_{n-2} & \lambda_{n-1} & w_{n-1} \\ & & & w_{n-1} & \lambda_n \end{pmatrix}.$$

## Proposition

Any eigenvalue of $J$ regarded as an operator in $\ell^2(\mathbb{N})$ is simple.

## Decomposition of a symmetric Jacobi matrix

Jacobi matrix $J_n$ can be decomposed into the product

$$J_n = G_n \tilde{J}_n G_n,$$

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- Next, $\tilde{\lambda}_k = \lambda_k / \gamma_k^2$ and

$$\gamma_{2k-1} = \prod_{j=1}^{k-1} \frac{w_{2j}}{w_{2j-1}}, \ \gamma_{2k} = w_1 \prod_{j=1}^{k-1} \frac{w_{2j+1}}{w_{2j}}, \ k = 1, 2, 3, \ldots .$$

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- Alternatively, sequence $\{\gamma_n\}$ can be defined recursively as $\gamma_1 = 1$, $\gamma_{k+1} = w_k / \gamma_k$.

# Characteristic function in terms of $\mathfrak{F}$

Let $n \in \mathbb{N}$ and $\{x_j\}_{j=1}^{n} \subset \mathbb{C}$ then one has

$$\mathfrak{F}(x_1, x_2, \dots, x_n) = \det \begin{pmatrix} 1 & x_1 & & & & \\ x_2 & 1 & x_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & x_{n-1} & 1 & x_{n-1} \\ & & & & x_n & 1 \end{pmatrix}.$$

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Let $n \in \mathbb{N}$ a $z \in \mathbb{C}$, then it holds

$$\det(J_n - zI_n) = \left( \prod_{k=1}^{n} (\lambda_n - z) \right) \mathfrak{F}\left( \frac{\gamma_1^2}{\lambda_1 - z}, \frac{\gamma_2^2}{\lambda_2 - z}, \dots, \frac{\gamma_n^2}{\lambda_n - z} \right).$$

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- Q: What one can say about the function $\mathfrak{F}\left(\left\{\frac{\gamma_k^2}{\lambda_k - z}\right\}_{k=1}^{\infty}\right)$?
- Q: Is this function related to the spectrum of $J$ somehow?

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- Let for at least one $z \in \mathbb{C} \setminus \operatorname{der}(\lambda)$ it holds

$$\sum_{n=1}^{\infty} \frac{w_n^2}{|\lambda_n - z||\lambda_{n+1} - z|} < \infty.$$

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## Proposition

The function

$$F_J(z) := \mathfrak{F}\left(\left\{\frac{\gamma_k^2}{\lambda_k - z}\right\}_{k=1}^{\infty}\right)$$

is analytic on $\mathbb{C} \setminus \overline{\lambda}$ and it has poles in points $z \in \lambda \setminus \mathrm{der}(\lambda)$ of order $r_z = \sum_{n=1}^{\infty} \delta_{(\lambda_n, z)} < \infty$. Moreover, all zeros of the function $F_J(z)$ are simple.

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- Then, by using the recurrence rule for the function $\mathfrak{F}$, one finds out the equation

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If

$$\xi_0(z) \equiv F_J(z) \equiv \mathfrak{F} \left( \left\{ \frac{\gamma_k^2}{\lambda_k - z} \right\}_{k=1}^{\infty} \right) = 0$$

for some $z \in \mathbb{C} \setminus \overline{\lambda}$, then $z$ is an eigenvalue of $J$ and vector $\xi(z) \equiv \{\xi_k(z)\}_{k=1}^{\infty}$ is the respective eigenvector.

**Theorem**

Let $J$ be self-adjoint. Then it holds

$$\mathfrak{Z}(J) = \mathrm{spec}_p(J) \setminus \mathrm{der}(\lambda)$$

where $\mathfrak{Z}(J)$ denotes a union of the set of all zeros of $F_J(z)$ with set

$$\left\{ z \in \lambda \setminus \mathrm{der}(\lambda) : \lim_{z' \to z} (z - z')^{r_z} F_J(z') = 0 \right\}.$$

**Proposition**

Let $\lim_{n\to\infty} w_n = 0$ then every accumulation point of $\lambda$ belongs to the essential spectrum of $J$, i.e.

$$\mathrm{der}(\lambda) \subset \mathrm{spec}_{ess}(J).$$

# Example 1 (unbounded operator)

- Let $\lambda_n = \alpha n$, $\alpha \neq 0$ and $w_n = w > 0$, $n = 1, 2, \ldots$. With this choice one has

$$J = \begin{pmatrix} \alpha & w & & & \\ w & 2\alpha & w & & \\ & w & 3\alpha & w & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \qquad \gamma_n = \begin{cases} 1, & \text{if } n \text{ odd} \\ w, & \text{if } n \text{ even.} \end{cases}$$

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- Since the term $(w/\alpha)^{\frac{z}{\alpha}}\Gamma(1 - z/\alpha)$ does not effect zeros of $F_J(z)$ and, moreover, the term $\Gamma(1 - z/\alpha)$ causes singularities in $z = \alpha, 2\alpha, \dots$, one arrives at the following expression for the spectrum

  and the formula for the $k$th entry of the respective eigenvector

## Example 1 (unbounded operator)

- Let $\lambda_n = \alpha n$, $\alpha \neq 0$ and $w_n = w > 0$, $n = 1, 2, \ldots$. With this choice one has

$$J = \begin{pmatrix} \alpha & w & & & \\ w & 2\alpha & w & & \\ & w & 3\alpha & w & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \qquad \gamma_n = \begin{cases} 1, & \text{if } n \text{ odd} \\ w, & \text{if } n \text{ even.} \end{cases}$$

- The characteristic function can be expressed as

$$F_J(z) = \left(\frac{w}{\alpha}\right)^{\frac{z}{\alpha}} \Gamma\left(1 - \frac{z}{\alpha}\right) J_{-\frac{z}{\alpha}}\left(\frac{2w}{\alpha}\right).$$

- Since the term $(w/\alpha)^{\frac{z}{\alpha}}\Gamma(1 - z/\alpha)$ does not effect zeros of $F_J(z)$ and, moreover, the term $\Gamma(1 - z/\alpha)$ causes singularities in $z = \alpha, 2\alpha, \ldots$, one arrives at the following expression for the spectrum

$$\operatorname{spec}(J) = \{z \in \mathbb{R};\ J_{-\frac{z}{\alpha}}\left(\frac{2w}{\alpha}\right) = 0\}$$

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$$v_k(z) = (-1)^k J_{k-\frac{z}{\alpha}}\left(\frac{2w}{\alpha}\right).$$

## Example 2 (compact operator)

- Let $\lambda_n = 1/n$ and $w_n = 1/\sqrt{n(n+1)}$, $n = 1, 2, \dots$. Then matrix J has the form

$$J = \begin{pmatrix} 1 & 1/\sqrt{2} & & & \\ 1/\sqrt{2} & 1/2 & 1/\sqrt{6} & & \\ & 1/\sqrt{6} & 1/3 & 1/\sqrt{12} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}. \qquad (1)$$

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$$F_J(z) = \sum_{s=0}^{\infty} \frac{1}{z^s} \frac{1}{s!} \prod_{j=1}^{s} \frac{1}{1 - jz} = z^{-\frac{1}{z}} \Gamma\left(1 - \frac{1}{z}\right) J_{-\frac{1}{z}}\left(\frac{2}{z}\right).$$

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## Example 3 (compact operator with zero diagonal)

- Let $\lambda_n = 0$, $w_n = \beta/\sqrt{(n+\alpha)(n+\alpha+1)}$, $\alpha > -1$, $\beta > 0$, $n = 1, 2, \ldots$. Then the results are

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$$v_k(z) := q^{\frac{(k-1)(k-2)}{2}} \left(\frac{\alpha}{z}\right)^k {}_0\phi_1\left(;0;q^2,-q^{2k+1}\left(\frac{\alpha}{z}\right)^2\right).$$

Thank you!