The Characteristic Function for Complex Jacobi Matrices

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Introduction

For $\lambda \equiv \{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{C}$ and $w \equiv \{w_n\}_{n=1}^{\infty} \subset \mathbb{C} \setminus \{0\}$, let us denote

$$\mathcal{J} := \begin{pmatrix} \lambda_1 & w_1 & & & \\ w_1 & \lambda_2 & w_2 & & \\ & w_2 & \lambda_3 & w_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Next, we denote by J an operator acting on $\ell^2(\mathbb{N})$ by formal matrix product $Jx := \mathcal{J}x$ with domain $\mathrm{Dom}(J) = \{x \in \ell^2(\mathbb{N}) : \mathcal{J}x \in \ell^2(\mathbb{N})\}$.
Further, we restrict ourself only on a particular class of Jacobi operators for which there exists $z_0 \in \mathbb{C} \setminus \mathrm{der}(\lambda)$ such that

$$\sum_n \left| \frac{w_n^2}{(\lambda_n - z_0)(\lambda_{n+1} - z_0)} \right| < \infty. \qquad (1)$$

$\mathrm{der}(\lambda)$ stands for the set of finite accumulation points of the sequence λ.

Function $\mathfrak{F}$

Definition 1 *Let us define $\mathfrak{F} : D \to \mathbb{C}$ by the relation*

$$\mathfrak{F}(x) = 1 + \sum_{m=1}^{\infty} (-1)^m \sum_{k_1=1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \ldots \sum_{k_m=k_{m-1}+2}^{\infty} x_{k_1}x_{k_1+1}x_{k_2}x_{k_2+1}\ldots x_{k_m}x_{k_m+1}$$

where

$$D = \left\{ \{x_k\}_{k=1}^{\infty} \subset \mathbb{C};\ \sum_{k=1}^{\infty} |x_k x_{k+1}| < \infty \right\}.$$

For a finite number of complex variables we identify $\mathfrak{F}(x_1, x_2, \ldots, x_n)$ with $\mathfrak{F}(x)$ where $x = (x_1, x_2, \ldots, x_n, 0, 0, 0, \ldots)$. Note the domain D is not a linear space. One has, however, $\ell^2(\mathbb{N}) \subset D$.

Properties of $\mathfrak{F}$

First, the function $\mathfrak{F}$ is continuous on $\ell^2(\mathbb{N})$ and for any $x \in D$ one has an estimate

$$|\mathfrak{F}(x)| \leq \exp\left(\sum_{k=1}^{\infty} |x_k x_{k+1}| \right).$$

Moreover, for $k = 1, 2, \ldots$, this function obey a three-term recurrence relation

$$\mathfrak{F}(x) = \mathfrak{F}(x_1, \ldots, x_k)\, \mathfrak{F}(T^k x) - \mathfrak{F}(x_1, \ldots, x_{k-1}) x_k x_{k+1} \mathfrak{F}(T^{k+1} x)$$

and, furthermore, limit relations

$$\lim_{n\to\infty} \mathfrak{F}(T^n x) = 1, \qquad \lim_{n\to\infty} \mathfrak{F}(x_1, x_2, \ldots, x_n) = \mathfrak{F}(x),$$

hold for any $x \in D$. T stands for the truncation operator from the left, $T(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$.
Second, variety classes of special functions are expressible in terms of $\mathfrak{F}$, first of all the Bessel function of the first kind,

$$J_\nu(2w) = \frac{w^\nu}{\Gamma(\nu+1)} \mathfrak{F}\left(\left\{ \frac{w}{\nu+k} \right\}_{k=1}^{\infty} \right)$$

where $\nu \notin -\mathbb{N}$ and $w \in \mathbb{C}$. Other classes of special function which relates $\mathfrak{F}$ (applied on a suitable sequence) involve, for instance, confluent hypergeometric functions and their q-analogues.

Fundamental references

[1] F. Štampach, P. Šťovíček: *On the eigenvalue problem for a particular class of finite Jacobi matrices*, Lin. Alg. App., 434, (2011), 1336-1353.

[2] F. Štampach, P. Šťovíček: *The characteristic function for Jacobi matrices with applications*, arXiv:1201.1743, (preprint).

Characteristic Function

We introduce the (renormalized) characteristic function associated with a Jacobi matrix $\mathcal{J}$ as

$$F_{\mathcal{J}}(z) := \mathfrak{F}\left(\left\{ \frac{\gamma_n^2}{\lambda_n - z} \right\}_{n=1}^{\infty} \right)$$

where sequence $\{\gamma_k\}_{k=1}^{\infty}$ is defined recursively by $\gamma_1 = 1,\ \gamma_{k+1} = w_k/\gamma_k$.
$F_{\mathcal{J}}$ is treated as a complex function of a complex variable z and, assuming (1), it is well defined on $\mathbb{C} \setminus \overline{\lambda}$ where $\overline{\lambda}$ denotes the closure of λ.
Moreover, $F_{\mathcal{J}}$ is analytic on $\mathbb{C} \setminus \overline{\lambda}$ and it has poles in points $z \in \lambda \setminus \mathrm{der}(\lambda)$ of finite order less or equal to

$$r(z) := \sum_{k=1}^{\infty} \delta_{\lambda_k, z}.$$

Spectrum and the Zero Set

Let us define

$$\mathfrak{Z}(\mathcal{J}) := \left\{ z \in \mathbb{C} \setminus \mathrm{der}(\lambda);\ \lim_{u\to z} (u-z)^{r(z)} F_{\mathcal{J}}(u) = 0 \right\}$$

and, for $k \in \mathbb{Z}_+$ and $z \in \mathbb{C} \setminus \mathrm{der}(\lambda)$, put

$$\xi_k(z) := \lim_{u\to z} (u-z)^{r(z)} \left(\prod_{l=1}^{k} \frac{w_{l-1}}{u - \lambda_l} \right) \mathfrak{F}\left(\left\{ \frac{\gamma_l^2}{\lambda_l - u} \right\}_{l=k+1}^{\infty} \right)$$

where one sets $w_0 = 1$. Note that for $z \notin \overline{\lambda}$, $\xi_0(z) = F_{\mathcal{J}}(z)$. Then the zero set $\mathfrak{Z}(\mathcal{J})$ coincide with the spectrum of J with the exception of points from $\mathrm{der}(\lambda)$:

Theorem 2 *Let condition (1) holds then one has*

$$\mathrm{spec}(J) \setminus \mathrm{der}(\lambda) = \mathrm{spec}_p(J) \setminus \mathrm{der}(\lambda) = \mathfrak{Z}(\mathcal{J}).$$

Moreover, for $z \in \mathfrak{Z}(\mathcal{J})$, $\xi(z) := \{\xi_k(z)\}_{k=1}^{\infty}$ is the corresponding eigenvector to the eigenvalue z.

Green Function

The Green function for the spectral parameter z,

$$G(z; i, j) := \langle e_i, (J - z)^{-1} e_j \rangle, \quad i, j \in \mathbb{N},$$

(a matrix in the standard basis) is given by the formula

$$G(z; i, j) = -\frac{1}{w_{\max(i,j)}} \left(\prod_{l=\min(i,j)}^{\max(i,j)} \frac{w_l}{z - \lambda_l} \right) \mathfrak{F}\left(\left\{ \frac{\gamma_l^2}{\lambda_l - z} \right\}_{l=1}^{\min(i,j)-1} \right)$$
$$\times \mathfrak{F}\left(\left\{ \frac{\gamma_l^2}{\lambda_l - z} \right\}_{l=\max(i,j)+1}^{\infty} \right) \mathfrak{F}\left(\left\{ \frac{\gamma_l^2}{\lambda_l - z} \right\}_{l=1}^{\infty} \right)^{-1}.$$

In particular, for the Weyl m-function one has

$$m(z) = \frac{1}{\lambda_1 - z} \mathfrak{F}\left(\left\{ \frac{\gamma_l^2}{\lambda_l - z} \right\}_{l=2}^{\infty} \right) \mathfrak{F}\left(\left\{ \frac{\gamma_l^2}{\lambda_l - z} \right\}_{l=1}^{\infty} \right)^{-1}.$$

Example

Let us set $\lambda_n = 1/n$, $w_n = \beta/\sqrt{n(n+1)}$, for all $n \in \mathbb{N}$, where $\beta \in \mathbb{C}$. By using Theorem 2, one can find out

$$\mathrm{spec}(J) = \{1/z;\ J_{-z}(2\beta z) = 0\} \cup \{0\}$$

and for the corresponding eigenvectors $v(z) = \{v_k(z)\}_{k=1}^{\infty}$ one has

$$v_k(z) = \sqrt{k}\, z^{-1/z} J_{k-1/z}\left(\frac{2\beta}{z} \right).$$