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presented:

Minimal dominating sets in interval graphs and trees

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Definitions

- Let $S \subseteq V$ and $v \in S$. We say the x is the **private neighbor** of v with respect to S, if no other vertex in S has x as its neighbor and x itself is not in S.
- $D \subseteq V$ is a **dominating set** if every vertex in G is either in D or is a neighbor of some vertex of D. We say that a vertex in D is dominating its neighbors.
- A dominating set D is **minimal**, if we can't remove any vertex from it without it ceasing to be a dominating set.
- A graph $G = (V, E)$ is an **interval graph**, if and only if there is a collection l of intervals of the real line and a bijection between V and l such that to each vertex $v \in V$ corresponds an interval $I_v = [l(v), r(v)] \in l$ and two vertices u and w are adjacent in G if and only if $I_u \cap I_w \neq \emptyset$.
- $l(v)$ is the left coordinate of the interval belonging to $v \in V$.
- $r(v)$ is the right coordinate of the interval belonging to $v \in V$.
- $O^*(f(n)) = O(f(n) \cdot n^{O(1)})$
- $\mu(G)$ is the number of minimal dominating sets in an interval graph G.
- $\mu(n)$ is $\max\{\mu(G) : |V(G)| = n\}$
- A vertex set $S \subseteq V$ is an **irredundant set** of G if every vertex of S has a private neighbor with respect to S.
- An irredundant set $U = \{u_1, u_2, \ldots, u_k\}$ in an interval graph is **good** if every vertex w satisfying $r(w) < l(u_i)$ has a neigbor in U.

Theorems

Whenever we mention a set of vertcies indexed with consecutive integers, like $U = \{u_1, u_2, \ldots, u_k\}$, it should be understood that $l(u_i) < l(u_{i+1})$ for $1 \leq i \leq k - 1$.

Theorem 1. Let G be an interval graph, and let $D = \{u_1, u_2, \ldots, u_k\}$ be a set of vertices in G. Then D is a minimal dominating set of G if and only if the following conditions hold:
1. $l(u_1) < r(v)$ for every vertex v of G.
2. $l(v) < r(u_k)$ for every vertex v of G.
3. for every $i \in [1, k-1]$: if $l(v) < r(u_i)$ for every vertex v of G, then $i = k$, otherwise $l(w) < r(u_{i+1})$, where w is the vertex with the smallest $l(w)$ such that $l(w) > r(u_i)$.
4. for every $i \in [1, k-1]$: $r(y) < l(u_{i+1}) < r(z)$, where y is the private neighbor of u_i with the smallest $r(y)$, and z is the vertex with the smallest $r(z)$ such that $l(z) > r(u_i)$

Corollary 1. Let $D = \{u_1, u_2, \ldots, u_k\}$ be a minimal dominating set of an interval graph. Then $\{u_1, \ldots, u_i\}$ is a good irredundant set for every $1 \leq i \leq k$.

Lemma 1. Let $U = \{u_1, \ldots, u_i\}$ be a good irredundant set of an interval graph, and let W be the set of vertices w satisfying $r(y) < l(w) < r(z)$, where y is the private neighbor of u_i with the smallest $r(y)$, and z is the vertex with the smallest $r(z)$ such that $l(z) > r(u_i)$. Then every good irredundant set that contains U contains at most one vertex of W.

Lemma 2. Let v be the vertex with the smallest $r(v)$ in an interval graph, and let X be the set of vertices x satisfying $l(x) < r(v)$. Then there is no good irredundant set that contains more than one vertex of X.

Theorem 2. There is a branching algorithm to enumerate all minimal dominating sets of an interval graph in time $O^*(3^{n/3})$.

Theorem 3. The maximum number of minimal dominating sets in an interval graph on n vertices is $3^{n/3}$.

Theorem 4. There is a branching algorithm to enumerate all minimal dominating sets of a tree in time $O^*(3^{n/3})$.

Theorem 5. The maximum number of minimal dominating sets in a forest on n vertices is at most $3^{n/3}$.