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presented:

# Decompositions of complete graphs into bipartite 2-regular subgraphs


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http://www.combinatorics.org/ojs/index.php/eljc/article/view/v23i2p1


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## Definitions

- A *decomposition* of a graph $K$ is a set $\{G_1, G_2, \ldots, G_t\}$ of subgraphs of $K$ such that $E(G_1) \cup E(G_2) \cup \cdots \cup E(G_t) = E(K)$ and $E(G_i) \cap E(G_j) = \emptyset$ for $1 \le i < j \le t$.
- Let $G$ be a graph and $\mathcal{D} = \{G_1, G_2, \ldots, G_t\}$ a decomposition such that $G_i$ is isomorphic to $G$, then $\mathcal{D}$ is called a *$G$-decomposition*.
- The complete graph of order $n$ is denoted by $K_n$, the cycle of order $n$ is denoted by $C_n$ and the path of order $n$ is denoted by $P_n$ (so $P_n$ has $n-1$ edges).
- If $n$ is odd, then $K_n^*$ is $K_n$, otherwise it is $K_n - I$, where $I$ denotes the edges of a perfect matching on $K$.
- Let $G$ be a 2-regular graph of order $k$ and $K_n^*$ defined as above. If there exists a $G$-decomposition of $K_n^*$ $(n \ge 3)$, then it is obvious that $3 \le k \le n$ and $k \mid n\left\lfloor \frac{n-1}{2} \right\rfloor$. These conditions are called *the obvious and neccessary conditions* for the existence of a $G$-decomposition of $K_n^*$.
- For a given graph $K$, we define the graph $K^{(2)}$ by $V(K^{(2)}) = V(K) \times \mathbb{Z}_2$
  and $E(K^2) = \{\{(x,a),(y,b)\} : \{x,y\} \in E(K), a, b \in \mathbb{Z}_2\}$.
- For each even $r \ge 2$ let $Y_r$ denote any graph isomorphic to the graph with vertex set $\{v_1, v_2, \ldots, v_{r+1}\}$ and edge set

$$\{(v_i, v_{i+1}) : i = 1, 2, \ldots, r\} \cup \{(v_1, v_3) \cup \{(v_i, v_{i+3}) : i = 2, 4, \ldots, r-2\}\}$$

  and specially $E(Y_2) = \{(v_1, v_2), (v_2, v_3), (v_1, v_3)\}$.
- Let $X_{2r}$ denote the graph obtained from $Y_r^{(2)}$ by adding the edges
  $\{((v_1, 0), (v_1, 1)), ((v_2, 0), (v_2, 1)), \ldots, ((v_{r+1}, 0), (v_{r+1}, 1))\}$.
- For each even $r \ge 2$ we define the graph $J_{2r}$ to be the graph with vertex set $V(J_{2r}) = \{u_1, u_2, \ldots, u_{r+2}\} \cup \{v_1, v_2, \ldots, v_{r+2}\}$ and the edge set

$$\begin{aligned} E(J_{2r}) = \{&(u_i, v_i) : i = 3, 4, \ldots, r+2\} \cup \\ &\{(u_i, u_{i+1}), (v_i, v_{i+1}), (u_i, v_{i+1}), (v_i, u_{i+1}) : i = 2, 3, \ldots, r+1\} \cup \\ &\{(u_i, u_{i+3}), (v_i, v_{i+3}), (u_i, v_{i+3}), (v_i, u_{i+3}) : i = 1, 3, \ldots, r-1\} \end{aligned}$$

### Problems

**Problem 1:** For each 2-regular graph $G$ and each positive integer $n$ satisfying the obvious necessary conditions, determine whether there exists a $G$-decomposition of $K_n^*$.

## Theorems

**Theorem 1.** If $G$ is a bipartite 2-regular graph of order $2m$, then there is a $G$-decomposition of $P_{m+1}^{(2)}$.
**Theorem 2.** If $r$ and $a$ are even where $r \le 2a-2, r \le 2b$ and $r \mid ab$, then there exists a $P_{r+1}$ decomposition of $K_{a,b}$.
**Theorem 3.** There is a $P_{r+1}$ decomposition of $K_v$ if and only if $v \ge r+1$ and $r \mid \frac{v(v-1)}{2}$.
**Lemma 4.** For each even $r \ge 2$, there exists a decomposition of $K_{r+1}$ into $\frac{r-2}{2}$ Hamilton paths and a copy of $Y_r$.
**Lemma 5.** If $r$ is even, $2 \le r \le \frac{m-1}{2}$, and $r \mid \frac{1}{2}m(m-1) - \frac{3r}{2}$, then there is a $P_{r+1}$-decomposition of $K_m - Y_r$.

**Lemma 6.** If $G$ is a bipartite 2-regular graph of order $2r$ where $r \geq 4$ is even, then there is a decomposition $\{H_1, H_2, H_3, H_4\}$ of $J_{2r}$ such that

(1) $V(H_1) = \{u_1, u_2, \ldots, u_r\} \cup \{v_3, v_4, \ldots, v_{r+2}\}$,

(2) $V(H_2) = \{u_3, u_4, \ldots, u_{r+2}\} \cup \{v_1, v_2, \ldots, v_r\}$,

(3) $V(H_3) = \{u_3, u_4, \ldots, u_{r+2}\} \cup \{v_3, v_4, \ldots, v_{r+2}\}$,

(4) $V(H_4) = \{u_3, u_4, \ldots, u_{r+2}\} \cup \{v_3, v_4, \ldots, v_{r+2}\}$,

(5) each of $H_1, H_2, H_3$ is isomorphic to $G$,

(6) $H_4$ is a 1-regular graph of order $2r$.

**Lemma 7.** If $r \geq 2$ is even and $G$ is any bipartite 2-regular graph of order $2r$, then there is a decomposition of $X_{2r}$ into three copies of $G$ and a 1-factor.

**Lemma 8.** If $n \geq 6$ is even and $G$ is any bipartite 2-regular graph of order $n-2$, then there is a $G$-decomposition of $K_n - I$.

**Lemma 9.** Let $r \geq 2$. If there is a $P_{r+1}$-decomposition of $K_m$ if $r$ is even and there is a $P_{r+1}$-decomposition of $K_m - Y_r$, then there is a $G$-decomposition of $K_{2m} - I$ for every bipartite 2-regular graph of order $2r$.

**Theorem 10.** Let $G$ be a bipartite 2-regular graph, let $k$ be the order of $G$, and let $n \geq 4$ be even. There exists a $G$-decomposition of $K_n - I$ if and only if $3 \leq k \leq n$ and $k \mid \frac{n(n-2)}{2}$, except possibly when $\frac{n}{2} < k < n-2$ and $\frac{n(n-2)}{2k}$ is odd both hold.