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presented:

Waiter-Client and Client-Waiter colourability games on a k-uniform hypergraph and the k-SAT game

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https://arxiv.org/pdf/1607.02258.pdf

Positional Games

Two-player perfect information games where each player takes turns to claim free elements of a set X until all members of X have been claimed. The winner is determined by a set $\mathcal{F}$ of so-called "winning sets".

Biased (1 : q) Waiter-Client game: Waiter offers exactly $q+1$ free elements of X to Client. Client claims one of the offered elements. The remaining elements are claimed by Waiter. If only $1 \leq r \leq q$ free elements remain in the last round, Waiter claims them all. Waiter wins the game if he can force the Client to fully claim a winning set in $\mathcal{F}$.
Biased (1 : q) Client-Waiter game: Waiter offers $1 \leq t \leq q+1$ free elements of X to Client. Client wins if he can fully claim a winning set in $\mathcal{F}$.

Definitions

- traversal family of $\mathcal{F}$ is $\mathcal{F}^* := \{A \subseteq X : A \cap B \neq \emptyset \text{ for every } B \in \mathcal{F}\}$
- traversal game: $(X, \mathcal{F}^*)$

Hypergraph

- k-uniform hypergraph: a generalization of a graph where each edge connects exactly k vertices
- independent set A: a set of vertices in a graph $\mathcal{H}$ such that $\{e \in E(\mathcal{H}) : e \subseteq A\} = \emptyset$
- independence number $\alpha(\mathcal{H})$: the maximum size of an independent set of vertices in a graph $\mathcal{H}$
- l-clique: a subgraph with l vertices in which every set of k vertices is an edge
- clique number $\omega(\mathcal{H})$: the largest l such that graph $\mathcal{H}$ contains an l-clique
- weak chromatic number $\chi(\mathcal{H})$: the smallest integer k for which vertices in graph $\mathcal{H}$ can be partitioned into k independent sets

k-SAT

- k-clause: the disjunction of exactly k non-complementary literals taken from n fixed boolean variables
- k-CNF boolean formula: the conjunction of any number of k-clauses
- $C_n^{(k)}$: a set of all possible k-clauses

Useful Tools

Lemma 2.2 Let $k \geq 2$ be an integer. Any k–CNF boolean formula in which no variable appears in more than $2^{k-2}/k$ k–clauses is satisfiable.

Theorem 2.3 Let q be a positive integer, let X be a finite set and let $\mathcal{F}$ be a family of subsets of X. If

$$\sum_{A \in \mathcal{F}} \left(\frac{q}{q+1}\right)^{|A|} < 1,$$

then Client has a winning strategy for the $(1:q)$ Client–Waiter traversal game $(X, \mathcal{F}^*)$.

Theorem 2.4 Let q be a positive integer, let X be a finite set and let $\mathcal{F}$ be a family of subsets of X. If

$$\sum_{A \in \mathcal{F}} 2^{-|A|/(2q-1)} < 1/2\,,$$

then Waiter has a winning strategy for the $(1:q)$ Waiter–Client traversal game $(X, \mathcal{F}^*)$.

Theorem 2.5 Let q be a positive integer, let X be a finite set, let $\mathcal{F}$ be a family of (not necessarily distinct) subsets of X and let $\Phi(\mathcal{F}) = \sum_{A \in \mathcal{F}} (q+1)^{-|A|}$. Then, when playing the $(1:q)$ Waiter–Client game $(X, \mathcal{F})$, Client has a strategy to avoid fully claiming more than $\Phi(\mathcal{F})$ sets in $\mathcal{F}$.

Results

Theorem 1.1 Let k, q and n be positive integers, with n sufficiently large and $k \geq 2$ fixed, and consider the $(1:q)$ Waiter–Client non–2–colourability game played on the edge set of the complete k–uniform hypergraph $K_n^{(k)}$ on n vertices. If $q \leq \binom{\lceil n/2 \rceil}{k} \frac{\ln 2}{2((1+\ln 2)n+\ln 2)}$, then Waiter can force Client to build a non–2–colourable hypergraph. Also, if $q \geq 2^{k/2} e^{k/2+1} k \binom{n}{k}/n$, then Client can keep his hypergraph 2–colourable throughout the game.

Theorem 1.5 Let k, q and n be positive integers, with n sufficiently large and $k \geq 2$ fixed, and consider the $(1:q)$ Client–Waiter k–SAT game played on $\mathcal{C}_n^{(k)}$. When $q < \binom{n}{k}/n$, Client can ensure that the conjunction of all k–clauses he claims by the end of the game is not satisfiable. However, when $q \geq 16k^3 \binom{n}{k}/n$, Waiter can ensure that the conjunction of all k–clauses claimed by Client is satisfiable throughout the game.