A probabilistic version of the game of Zombies and Survivors on graphs

Abstract

We consider a new probabilistic graph searching game played on graphs, inspired by the familiar game of Cops and Robbers. In Zombies and Survivors, a set of zombies attempts to eat a lone survivor loose on a given graph. The zombies randomly choose their initial location, and during the course of the game, move directly toward the survivor. At each round, they move to the neighboring vertex that minimizes the distance to the survivor; if there is more than one such vertex, then they choose one uniformly at random. The survivor attempts to escape from the zombies by moving to a neighboring vertex or staying on his current vertex. The zombies win if eventually one of them eats the survivor by landing on their vertex; otherwise, the survivor wins. The zombie number of a graph is the minimum number of zombies needed to play such that the probability that they win is at least 1/2. We present asymptotic results for the zombie numbers of several graph families, such as cycles, hypercubes, incidence graphs of projective planes, and Cartesian and toroidal grids.

Introducton

In **Section 2**, we give an example of a sequence of graphs $(G_n)_{n\in\mathbb{N}}$ having $Z(G_n) = \Theta(n)$.
In **Section 3**, we discuss cycle graphs. Theorem 3.3 gives the asymptotic value of the zombie number of cycles.
In **Section 4**, we consider the zombie number of the incidence graphs of projective planes. By using double exposure and coupon collector problems, we show in Theorem 4.1 that about two times more zombies are needed to eat the survivor than cops.
In **Section 5**, we consider hypercubes Qn and show in Theorem 5.1 that $z(Q_n) \sim \frac{2}{3}n$, as $n \to \infty$.
In **Section 6**, we consider both Cartesian grids and grids formed by products of cycles (so called toroidal grids). In toroidal grids, we prove in Theorem 6.2 a lower bound for the zombie number of $\sqrt{n}/(\omega \log n)$, where $\omega = \omega(n)$ is going to infinity as n → ∞. The proof relies on the careful analysis of a strategy for the survivor.

Cartesian product of G and H, written G X H, to have vertices V(G) × V(H), and vertices (a, b) and (c, d) are joined if *a = c* and *bd∈E(H)* or *ac∈E(G)* and *b = d*.

The notations **o(·)** and **O(·)** refer to functions of n, not necessarily positive, whose growth is bounded.

An event in a probability space holds **asymptotically almost surely** (or a.a.s.) if the probability that it holds tends to 1 as n goes to infinity.

$$f(n) \sim g(n) \text{ if } f(n)/g(n) \to 1 \text{ as } n \to \infty\text{; that is, when } f(n) = (1+o(1))g(n).$$

Section 2 - The cost of being undead can be high

Theorem 2.1. For Gn as defined above, we have

$$z(G_n) \sim \log(1/\alpha)n \approx 0.2180n,$$

where α is the only solution of $\alpha^4(2-\alpha) = 1/2$ in (0, 1). In particular, $Z(G_n) \sim \frac{\log(1/\alpha)}{2}n$.

Section 3 – Cycles

Lemma 3.1. The survivor wins on C_n against k ≥ 2 zombies if and only if all zombies are initially located on an induced subpath containing at most $\lceil n/2 \rceil - 2$ vertices.

Lemma 3.2. For any natural numbers k ≥ 2 and n ≥ 9, we have that

$$k\left(\frac{1}{2} - \frac{4}{n}\right)^{k-1} \le s_k(C_n) < k\left(\frac{1}{2}\right)^{k-1}$$

In particular, $s_k(C_n) \sim k(1/2)^{k-1}$, *as* $n \to \infty$.

Theorem 3.3.

$$z(C_n) = \begin{cases} 4 & \text{if } n \geq 27 \text{ or } n = 23, 25, \\ 3 & \text{if } 11 \leq n \leq 22 \text{ or } n = 9, 24, 26, \\ 2 & \text{if } 4 \leq n \leq 8 \text{ or } n = 10, \\ 1 & \text{if } n = 3; \end{cases}$$

and therefore,

$$Z(C_n) = \begin{cases} 2 & \text{if } n \geq 27 \text{ or } n = 23, 25, \\ 3/2 & \text{if } 11 \leq n \leq 22 \text{ or } n = 9, 24, 26, \\ 1 & \text{if } 3 \leq n \leq 8 \text{ or } n = 10. \end{cases}$$

Section 4 - Projective planes

An **incidence structure** consists of a set P of points, and a set L of lines along with an incidence relation consisting of ordered pairs of points and lines.

A **projective plane** consists of a set of points and lines satisfying the following axioms:

- There is exactly one line incident with every pair of distinct points.
- There is exactly one point incident with every pair of distinct lines.
- There are four points such that no line is incident with more than two of them.

Theorem 4.1. *$z(G_q) = 2q + \Theta(\sqrt{q})$, as $q \to \infty$. Hence, $Z(G_q) \sim 2$, as $q \to \infty$.*

Lemma 4.2. *Let $k = 2q + \omega\sqrt{q}$, where $\omega = \omega(q)$ is any function tending to infinity as $q \to \infty$. Then $s_k(G_q) \to 0$, as $q \to \infty$.*

Lemma 4.3. *Let $k = 2q - \omega\sqrt{q}$, where $\omega = \omega(q)$ is any function tending to infinity as $q \to \infty$. Then $s_k(G_q) \to 1$, as $q \to \infty$.*

Section 5 – Hybercubes

Theorem 5.1. *$z(Q_n) = \frac{2n}{3} + \Theta(\sqrt{n})$, as $n \to \infty$. Hence, $Z(Q_n) \sim \frac{4}{3}$, as $n \to \infty$.*

Lemma 5.2. *Let $k = \frac{2}{3}n - \omega\sqrt{n}$, where $\omega = \omega(n)$ is any function tending to infinity as $n \to \infty$. Then $s_k(Q_n) \to 1$, as $n \to \infty$.*

Lemma 5.3. *Let $k = \frac{2}{3}n + \omega\sqrt{n}$, where $\omega = \omega(n)$ is any function tending to infinity as $n \to \infty$. Then $s_k(Q_n) \to 0$, as $n \to \infty$.*

Section 6 – Grids

Theorem 6.1. *For $n \geq 2$, we have that $z(G_n) = 2$. Hence, $Z(G_n) = 1$.*

Theorem 6.2. *Let $\omega = \omega(n)$ be a function tending to infinity as $n \to \infty$. Then a.a.s. $z(T_n) \geq \sqrt{n}/(\omega \log n)$.*

Definition 6.3. A survivor strategy is B-boxed during the time period [0, 4n] if the following hold: the initial position u0 belongs to the box B and is chosen independently of the positions of the zombies; the sequence of moves $\boldsymbol{m} = (m_t)_{t\in[1,4n]}$ is such that the survivor always stays inside of B, regardless of the positions of the zombies in that period; each move m_t(t ∈ [1, 4n]) does not depend on the positions of the zombies that lie outside of B at that given step t (that is, any two configurations of the zombies at time t that only differ in the positions of some zombies not in B must yield the same value of m_t).

Lemma 6.4. *Assume that the survivor's strategy is B-boxed during the time period $[0, 4n]$, and pick any zombie strategy for all but one distinguished zombie. For any $t \in [1, 4n]$, the probability that this zombie is initially outside of the box B and arrives at B at the t-th step of the game is at most $20Kt \log n/n^2$.*

Lemma 6.5. *Consider $k = \lfloor \sqrt{n}/(\omega \log n) \rfloor$ zombies on T_n, for any given $\omega = \omega(n)$ that tends to infinity as $n \to \infty$. Assume that the survivor follows a B-boxed strategy during the time period $[0, 4n]$. Then a.a.s. the following hold:*

- there is no zombie in B initially;
- all zombies arrive to B within the first 3n steps; and
- no two zombies arrive at B less than M logn steps apart, where M = 12K

Definition 6.6. A zombie strategy $(v_0, \boldsymbol{\sigma})$ is called *regular* if, for any direction $\alpha \in \{\mathrm{L}, \mathrm{R}, \mathrm{U}, \mathrm{D}\}$ and any interval of consecutive steps $I \subseteq [1, 4n]$ of length $\lfloor 20 \log n \rfloor$, there is a subset of steps $J \subseteq I$ (not necessarily consecutive) with $|J| = \lceil \log n \rceil$ such that, for every $i \in J$, σ_i has α as the first symbol in the permutation. Informally, for every $\lfloor 20 \log n \rfloor$ consecutive steps in $[1, 4n]$, there are at least $\log n$ steps in which the zombie "tries" to move in the direction of α if that decreases the distance to the survivor.

Lemma 6.7. *Consider $k = \sqrt{n}/(\omega \log n)$ zombies on T_n, for any given $\omega = \omega(n)$ that tends to infinity as $n \to \infty$. Then a.a.s. every zombie has a regular zombie strategy.*

Definition 6.8. We call a trajectory to be *stable* if all its turning points are proper and, for any two different turning points j and j', we have $j - j' \geq \lfloor 20 \log n \rfloor$.

Lemma 6.9. *Suppose that a zombie has a fixed regular zombie strategy $(v_0, \boldsymbol{\sigma})$ and that the survivor follows a stable trajectory $\boldsymbol{u}$ during the time interval $[a, b]$. Let $\boldsymbol{v}$ be the trajectory of the zombie (determined by its strategy and the survivor's trajectory). Suppose moreover that v_a and u_a are at distance $d \in \{2, 3\}$ and that v_{a+1} and u_{a+1} are also at distance d (that is, the first move of the survivor is not towards the zombie). Then deterministically, v_t and u_t are at distance d for all $t \in [a, b]$.*