Jakub Vašíček

vasicja2@fit.cvut.cz

presented:

To catch a falling robber

William B. Kinnersley, Paweł Prałat, Douglas B. West

http://www.sciencedirect.com/science/article/pii/S0304397516001614

Definitions

Cops–and–Robber Game

- Pursuit–evasion game on a graph
- Players: one robber and some number of cops. The cops and then the robber choose their starting vertex. In each round the players can move to an adjacent vertex — the cops move in odd rounds and the robber in even rounds.
- The cops win if at some point there is a cop occupying the same vertex as the robber.
- The **cop number** of a graph $c(G)$ is the least number of cops that can guarantee winning (all players always know each others' positions).

  Variant of the game
- Each player holds their own set of elements. The cops start all with an empty set, the robber starts with a full set of n elements ($\{1, 2, ..., n\}$).
- Cops move by adding elements into their sets, the robber by removing elements from the set. The cops win if at some point a cop holds exactly the same set as the robber.

⇒ The cops have only one chance to win — in the round $n/2$ for even n and $n/2 + 1$ for odd n.
- c_n denotes the cop number of this graph.

Theorems

Theorem 1: The lower bound of c_n given by Alan Hill: $c_n \geq \begin{cases} 2^m, & n = 2m, \\ \binom{2m+1}{m+1} 2^{-m}, & n = 2m+1. \end{cases}$

Theorem 2: The upper bound of c_n given in this article: $c_n \leq \begin{cases} 2^m \ln n, & n = 2m, \\ 2^{-m} \binom{2m+1}{m+1} \ln n, & n = 2m+1. \end{cases}$

Strategies of the cops and the robber

Strategy of the robber used to prove the lower bound:

- Deleting an element from the set at the robber's current position evades all cops whose set contains that element.
- The robber strategy greedily evades as many cops as possible with each move (erases the most common element in cops' sets).

Strategy of the cops used to prove the upper bound:

- The strategy may or may not succeed in capturing the robber. However, with sufficiently many cops, the strategy succeeds *asymptotically almost surely* (or a.a.s.), that is, with probability tending to 1 as n tends to infinity. Consequently, some deterministic strategy for the cops (in response to the moves by the robber) wins the game.
- Let R be the current set occupied by the robber. On his k-th turn, for $1 \leq k \leq m$, each surviving cop at set C chooses the next element for his set uniformly at random from among $R \setminus C$. Regardless of how the robber moves, this cop strategy succeeds a.a.s.
- We say that the instance of the game satisfies property $P(t, a)$ if, after t rounds, every m-set below the robber also has at least a cops at or below it.
- The m-sets below the robber are the places where the robber can potentially be captured; property $P(t, a)$ means that each of them can be reached by at least a cops.
- To show that the cop strategy a.a.s. captures the robber, the article shows that, no matter how the robber plays, a.a.s. property $P(t_i, a_i)$ holds for specific choices of t_i and a_i.